Probe Ptolemaic Graphs

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Abstract. Given a class of graphs, $\mathcal{G}$, a graph G is a *probe graph of* $\mathcal{G}$ if its vertices can be partitioned into two sets, $\mathbb{P}$ (the probes) and $\mathbb{N}$ (the nonprobes), where $\mathbb{N}$ is an independent set, such that G can be embedded into a graph of $\mathcal{G}$ by adding edges between certain nonprobes. In this paper we study the probe graphs of ptolemaic graphs when the partition of vertices is unknown. We present some characterizations of probe ptolemaic graphs and show that there exists a polynomial-time recognition algorithm for probe ptolemaic graphs.

1 Introduction

In some applications, we want to determine the relation between every pair of elements in a set. If the relation is of boolean type, it can be described by a simple graph without self-loops. In some applications it is expensive to determine the relation between every pair of elements. Therefore, the elements are partitioned into *probes* and *nonprobes*. The relation between two elements is determined whenever at least one of the two elements is a probe. In graph-theoretical terms, we have a graph G whose vertices are partitioned into a set $\mathbb{P}$ of *probes* and a set $\mathbb{N}$ of *nonprobes*. The set of nonprobes $\mathbb{N}$ is an independent set. We want to know whether we can let G satisfy some property Π by adding edges between certain nonprobes. Let $\mathcal{G}$ be the class of graphs satisfying the property Π.

A graph G is a *probe graph of* $\mathcal{G}$ if its vertex set can be partitioned into a set $\mathbb{P}$ of *probes* and an *independent set* $\mathbb{N}$ of *nonprobes*, such that G can be embedded into a graph of $\mathcal{G}$ by adding edges between certain nonprobes. If the partition of

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the vertices of a graph G into a set of probes $\mathbb{P}$ and a set of nonprobes $\mathbb{N}$ is part of the input then we call G a *partitioned probe graph of* $\mathcal{G}$ if G can be embedded into a graph of $\mathcal{G}$ by adding edges between certain vertices of $\mathbb{N}$. We denote a partitioned graph as $G = (\mathbb{P}+\mathbb{N}, E)$, and when this notation is used it is to be understood that $\mathbb{N}$ is an independent set. We will refer to the class of (partitioned) probe graphs of the class of $\mathcal{G}$ graphs as (partitioned) probe $\mathcal{G}$ graphs.

For the partitioned case, there are efficient algorithms for the recognition problem on some classes of graphs, *e.g.*, probe interval graphs [18],[22], probe permutation graphs [5], probe distance-hereditary graphs [6], probe comparability graphs [7], and so on. In graph theory, the unpartitioned case is more interesting. Only few graph classes have polynomial-time recognition algorithms for their probe versions in unpartitioned case. These classes are chordal [3], interval [8], cograph [9,21], split [9],[20], P_4-reducible [9], and P_4-sparse graphs [9].

In this paper, we study the unpartitioned case of probe ptolemaic graphs. The remaining of this paper is organized as follows. Some notation and preliminaries are given in Section 2. We then propose some characterizations of probe ptolemaic graphs in Section 3. In Section 4, we consider the recognition of a special class of probe ptolemaic graphs, called *probe chordal cographs*. In Section 5, we show that there exists a polynomial-time algorithm that checks whether a graph is a probe ptolemaic graph. Finally, we give conclusion in the last section.

2 Preliminaries

A graph G is a pair (V, E), where the elements of V are called the *vertices* of G and where E is a family of two-element subsets of V, called the *edges*. We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. We write $n = |V|$ for the number of vertices and $m = |E|$ for the number of edges. We denote edges of a graph G as (x, y) (or xy) and we call x and y the end vertices of the edge. Unless stated otherwise, a graph is regarded as undirected. For a vertex x we write $N(x)$ for its set of neighbors in G, and for a subset $W \subseteq V$ we write $N(W) = \cup_{x \in W} N(x) - W$. For other conventions on graph-related notation we refer to any standard textbook. For graph classes not defined here we refer to [4].

For two sets A and B we write $A + B$ and $A - B$ instead of $A \cup B$ and $A \setminus B$ respectively. We write $A \subseteq B$ if A is a subset of B with possible equality and we write $A \subset B$ if A is a subset of B and $A \neq B$. For a set A and an element x we write $A - x$ instead of $A - \{x\}$ and $A + x$ instead of $A \cup \{x\}$.

For a graph $G = (V, E)$ and a subset $S \subseteq V$, we write $G[S]$ for the subgraph of G *induced* by S. For a subset $W \subseteq V$, we write $G-W$ for the graph $G[V-W]$, *i.e.*, the subgraph induced by $V-W$. For a vertex x we write $G-x$ rather than $G-\{x\}$.

For a partitioned probe graph $G = (\mathbb{P}+\mathbb{N}, E)$ of some graph class $\mathcal{G}$, an *embedding* of G is a graph of $\mathcal{G}$ obtained by adding edges between certain nonprobes, *i.e.*, vertices of $\mathbb{N}$.

Originally, ptolemaic graphs were defined as follows.

Definition 1 ([19]). *A connected graph is* ptolemaic *if for every four vertices x, y, u, and v:*

$$d(x, y)d(u, v) \leq d(x, u)d(y, v) + d(x, v)d(y, u)$$

A graph is ptolemaic *if every component is ptolemaic.*

Ptolemaic graphs can be characterized as those chordal graphs in which every 5-cycle has at least three chords, or, as those chordal graphs in which all chordless paths are shortest paths [4],[12],[27]. We will use the following characterization of ptolemaic graphs.

Theorem 1 ([17]). *A graph is ptolemaic if and only if it is distance hereditary and chordal.*

Recall that a graph is *chordal* if it has no induced cycle of length more than 3 and is *distance hereditary* if the distance between any two vertices remains the same in every connected induced subgraph. For the house, hole, domino, and gem, we refer to Fig. 1.

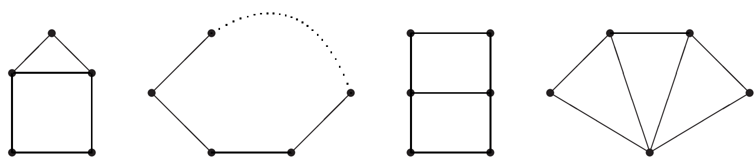

Fig. 1. A house, a hole, a domino, and a gem

Corollary 1. *Ptolemaic graphs are those chordal graphs which are gem-free.*

Definition 2 ([23]). *A* module *is a set M of vertices such that every vertex outside M is either adjacent to all vertices of M or to no vertex of M.*

A module which is also a clique is called a *clique module*. If $M = \varnothing$, $|M| = 1$, or M is the whole vertex set V, then M is called *trivial*.

One major characteristic, on which our recognition algorithms rely, is the following characterization of ptolemaic graphs. For similar characterizations, see also [24],[28].

Theorem 2 ([24],[28]). *A graph G is ptolemaic if and only if for every vertex x and for every component C of $G - N[x]$, $N(C)$ is a clique module in $G[C + N(C)]$.*

Proof. Since G is chordal, for every vertex x and every component C of $G - N[x]$, $N(C)$ is a clique. Assume some vertex $y \in C$ is adjacent to a vertex $\alpha \in N(C)$ and nonadjacent to a vertex $\beta \in N(C)$. Consider a β, y-path P with internal vertices in C. Then α is adjacent to every vertex of P, otherwise there is a chordless cycle of length at least 4 in G. Together with x this creates a gem.

Conversely, if G has a gem, then let x be one of the vertices of degree two. The neighborhood of x contains the universal vertex of the gem and the neighbor of x in the P_4 of the gem. The edge of the gem which does not intersect $N(x)$ is contained in some component C of $G - N[x]$. The other vertex of degree two contradicts the assumption that $N(C)$ is a clique module in the graph induced by $N[C]$. $\square$

It is well-known that chordal graphs can be characterized as those graphs in which every induced subgraph has a simplicial vertex [11],[14],[26].

Definition 3. *A vertex x is* simplicial *if its neighborhood $N(x)$ is a clique.*

We end this section with two other characterizations of ptolemaic graphs. Another way to characterize chordal graphs is by their minimal separators. A *minimal x,y-separator* for nonadjacent vertices x and y is a minimal set of vertices S such that x and y are in different components of $G-S$, the graph induced by $V-S$, where V is the set of vertices of G. A *minimal separator* is a set which is a minimal x,y-separator for some nonadjacent vertices x and y. A classic result of Dirac's says that a graph is chordal if and only if every minimal separator is a clique [11].

Theorem 3. *A connected graph is ptolemaic if and only if every connected induced subgraph either is a clique, or has a cutvertex, or has a separator which is a nontrivial clique-module.*

Proof. To see this, assume that G is ptolemaic and let H be a connected induced subgraph. Thus H is ptolemaic. If H is a clique or has a cutvertex, we are done. Otherwise, let x be a non-universal vertex. Let C be a component of $H-N[x]$ such that $N(C)$ is inclusion minimal. By Theorem 2 on the facing page, $S=N(C)$ is a clique module in $H[S+C]$. Since H is chordal, there exists a vertex $y \in C$ with S in its neighborhood. It follows, again by Theorem 2, that S is also a clique module in the component of $G-N[y]$ that contains x. Thus S is a separating clique module. To see the converse, notice that gems and chordless cycles of length at least 4 are not cliques, have no cutvertices, and have no nontrivial clique-modules. □

Theorem 4 ([17]). *A graph is G is ptolemaic if and only if for every pair of vertices x and y with $d(x,y)=2$, $N(x) \cap N(y)$ is a clique separator in G.*

3 Characterizations of Probe Ptolemaic Graphs

A graph is *HHD-free* if none of its induced subgraphs is a hole, a house, or a domino. HHD-free graphs can be recognized in $O(n^3)$ time ([16]).

A *ptolemaic twin* in a partitioned graph $G=(\mathbb{P}+\mathbb{N},E)$ is pair of vertices x,y such that

- $x,y \in \mathbb{P}$ and $N[x]=N[y]$, or
- $x,y \in \mathbb{N}$ and $N(x)=N(y)$, or
- $x \in \mathbb{P}, y \in \mathbb{N}$ and $N[x] \cap \mathbb{P} = N(y)$.

A *k-fan* is the graph consisting a k-path P_k and with a fully adjacent vertex. Thus a 4-fan is the gem. For a given graph $G=(V,E)$, a particular 2-CNF instance $F(G)$ is created as follows.

- The boolean variables are the vertices of G,
- for each edge ab of G, $(\overline{a} \vee \overline{b})$ is a clause, the *edge clause* for ab,
- for each $C_4 = abcd$ of G, $(a \vee b)$ and $(c \vee d)$ are two clauses, the C_4 *clauses* for that C_4,

– for each 4-fan with $P_4 = abcd$ of G, $(a \vee b)$, $(c \vee d)$ and $(\overline{a} \vee \overline{d})$ are three clauses, the 4-*fan clauses* for that 4-fan,
– for each 5-fan with $P_5 = abcde$ of G, $(\overline{b})$ and $(\overline{d})$ are two clauses, the 5-*fan clauses* for that 5-fan.

The formula $F(G)$ is the conjunction of all edge clauses, all C_4 clauses, all 4-fan clauses, and all 5-fan clauses. We will see that the recognition of probe ptolemaic graphs is 'in fact' a 2-SAT problem for 2-CNF instances like $F(G)$ defined above. Similar considerations have been made for other probe graph classes in [2],[20]; see also [25].

Finally, it is clear that a graph (partitioned or not) is probe ptolemaic if and only if each of its blocks (which can be detected in linear time) is probe ptolemaic. Moreover, probe ptolemaic graphs are HHD-free.

Theorem 5. *Let $G = (\mathbb{P} + \mathbb{N}, E)$ be a partitioned graph. Then the following statements are equivalent.*

(i) *G is a partitioned probe ptolemaic graph;*
(ii) *Each 2-connected induced subgraph of G has a ptolemaic twin;*
(iii) *G is HHD-free and $F(G)$ is satisfied by assigning nonprobe variables to* `true` *and probe variables to* `false`*;*
(iv) *G is HHD-free and $(F_1, \ldots, F_8)$-free; see Fig. 2.*

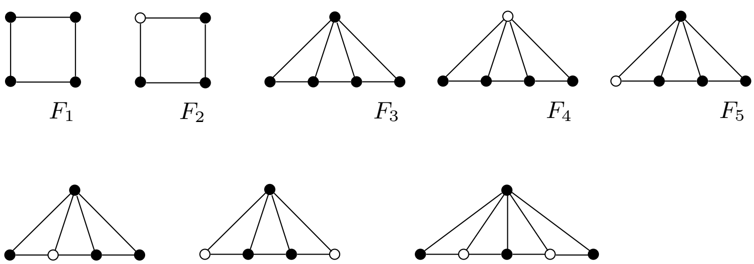

Fig. 2. Partitioned HHD-free graphs that are minimal but not probe ptolemaic; probes black, nonprobes white

Due to the limitation of space, we omit the proof of Theorem 5. For the unpartitioned case, by Theorem 5, we have the following theorem (for the same reason we omit the proof).

Theorem 6. *A graph G is a probe ptolemaic graph if and only if G is HHD-free and $F(G)$ is satisfiable.*

4 Probe Chordal Cographs

Assume G has a universal vertex, *i.e.*, a vertex x which is adjacent to all other vertices. Then, obviously, $G - x$ must be embedded into a chordal graph without an induced P_4, otherwise the embedding has a gem.

Definition 4 ([10],[13]). *A* cograph *is a graph without induced* P_4*.*

Chordal cographs can be characterized as follows. For a graph G let $\Upsilon = \Upsilon(G)$ be the set of *universal* vertices, *viz*, the set of vertices adjacent to all other vertices. Obviously, for any graph G, Υ induces a clique in G.

Proposition 1 ([29],[30]). *A connected graph is a chordal cograph if and only if it is complete or* $G - \Upsilon(G)$ *is a disconnected chordal cograph.*

Remark 1. Chordal cographs have the following tree-model: Let T be a rooted tree. Make any two vertices adjacent if they are contained in a common path from a leaf to the root.

From proposition 1 we easily obtain:

Lemma 1. *A partitioned graph* $G = (\mathbb{P} + \mathbb{N}, E)$ *can be embedded into a chordal cograph if and only if every connected induced subgraph has a nonempty set* Υ *of vertices that can be made universal by adding edges between nonprobes.*

The condition of Lemma 1 can be formulated in monadic second-order logic without edge-set quantifiers. Probe chordal-cographs have rankwidth at most 2. Therefore:

Corollary 2. *There exists an* $O(n^3)$ *algorithm which checks if a graph has an independent set* $\mathbb{N}$ *of nonprobes such that the partitioned graph* $G = (\mathbb{P} + \mathbb{N}, E)$ *can be embedded into a chordal cograph by adding some edges between nonprobes.*

For alternative algorithms see [6],[21].

5 Recognition of Probe Ptolemaic Graphs

In this section we show that there exists a polynomial-time recognition algorithm for the class of probe ptolemaic graphs. We assume that G is connected, otherwise we can test each component of G separately. By Section 4, we may assume that G has no universal vertices.

It is routine to check the following:

Proposition 2. *A probe ptolemaic graph has no induced house, hole, or domino. Furthermore, an induced gem has exactly two nonprobes, and they are at distance two in the* P_4 *of the gem.*

Lemma 2. *Consider two vertices* x *and* y *at distance 2 in* G*. Assume there exists an embedding of* G *in which* x *and* y *are probes. Let* H *be the graph obtained from* G *by adding the following edges:*

(i) *If* G *has an induced* C_4 *containing* x *and* y*, then the two nonprobes in the* C_4 *form an edge in* H*.*
(ii) *If* G *has a gem* Γ *with* x *and* y *at distance 2, then the two nonprobes in* Γ *form an edge in* H*.*

Then every gem or C_4 *in* H *induces also a gem or* C_4 *in* G*.*

Proof. By Theorem 4, the common neighborhood $S = N(x) \cap N(y)$ is a clique separator in any embedding. The claim follows from Proposition 2 and Theorem 2. □

Lemma 3. *Assume G is ptolemaic. Let x be a non-universal vertex and let C_1 and C_2 be two components of $G - N[x]$. Then either $N(C_1) \subseteq N(C_2)$, $N(C_2) \subseteq N(C_1)$, or $N(C_1) \cap N(C_2) = \varnothing$.*

Proof. Write $S = N(x)$, and $S_i = N(C_i)$, $i = 1, 2$. Recall that S_1 and S_2 are cliques. If $N(C_1) = S$ or $N(C_2) = S$ then the claim is obvious, so assume that both $N(C_i)$ are proper subsets of S. Let $\gamma \in S_1 \cap S_2$, $\alpha \in S_1 - S_2$, and $\beta \in S_2 - S_1$. Let $c_i \in C_i$, $i = 1, 2$, be a vertex in C_i adjacent to all vertices of S_i. Since G is chordal these vertices exist. First assume that α and β are not adjacent. Then we find a gem with universal vertex γ and P_4; $[c_1, \alpha, x, \beta]$. If α and β are adjacent, we find a gem with universal γ and a P_4, $[c_1, \alpha, \beta, c_2]$ in its neighborhood. □

Definition 5. *A* block *is a pair (x, C) where x is a vertex and C a component of $G - N[x]$.*

In our algorithm we first make a table of the blocks and order them according to increasing number of vertices in the component. For each block (x, C) we determine whether there exists an embedding of $G[x+N(C)+C]$ into a ptolemaic graph both in the case where x is a nonprobe and where x is a probe. When the type of x is fixed, the set A of vertices in $G[N[C] + x]$ which are at distance 2 from x in an embedding, is determined. We store in a table whether there exists an embedding of the block for both cases; with x of type 'probe' or 'nonprobe.' Furthermore, the table specifies if there exists such an embedding such that all vertices of A are nonprobes. We now describe how to obtain this information for each block.

Consider a block (x, C). Let $S = N(C)$. Suppose we want to check whether there exists an embedding of $G[x + S + C]$ with x as a probe. Notice that S is a clique in any embedding. Thus the partition of S is determined, except when it induces a clique in G in which case it can contain exactly one nonprobe.

We may add edges to destroy the induced 4-cycles and gems in $G[x + S + C]$ that contain x. This makes a clique of S and it makes S a module. Let $A \subseteq C$ be the set of vertices with a neighbor in S. If G has a nonedge between S and A then the partition into probes and nonprobes of $S + A$ is resolute. Otherwise, at most one of the two sets may contain nonprobes. Let $C_1, \ldots, C_t$ be the components of $C - A$.

Assume S has a probe u. Add edges to destroy C_4's that contain u and gems that contain u as a non-universal vertex. This makes a clique of each $S_i = N(C_i)$ and it makes S_i a module for $C_i + S_i$. Let $A_i \subseteq C_i$ be the set of vertices adjacent to S_i.

Now assume that some pair of minimal separators S_i and S_j *overlap*, *i.e.*, $S_i \cap S_j \neq \varnothing$, $S_i - S_j \neq \varnothing$, and $S_j - S_i \neq \varnothing$. By Lemma 3, in any embedding all vertices of A_i are adjacent to $S_j - S_i$ or all vertices of A_j are adjacent to $S_j - S_i$. Then, say, all vertices of A_i must be nonprobes. Thus S_i contains only probes.

By table look-up, we check if there exists an embedding for the components of $C_i - A_i$ such that A_i consists of only nonprobes.

Assume that S has only nonprobes. Consider all gems and C_4's in $S + C$ incident with vertices of S, and add the edges between the nonprobes. By Lemma 2, this fixates A. Notice that the partition of A into probes and nonprobes is determined: the set of probes in A is exactly $N(\sigma) \cap A$ for any $\sigma \in S$. Let $C_1, \ldots, C_t$ be the components of $C - A$. Then S_i is a module for C_i, otherwise there would be a gem incident with a vertex of S. Let A_i be the set of vertices in C_i adjacent to S_i. By table look-up we check if there exist embedding for blocks (u, C_i) where u is a nonprobe in S. If there are overlapping separators S_i and S_j, we check if there are embeddings such that A_i or A_j consists only of nonprobes.

The case where x is a nonprobe, is similar to the description above.

Theorem 7. *There exists an efficient algorithm to check whether a given graph $G = (V, E)$ is probe ptolemaic.*

Proof. For the case where there exists a universal vertex, the algorithm is described in Section 4. Otherwise, choose a vertex x such that the largest component C of $G - N[x]$ has maximum cardinality. Let $S = N(C)$. By the choice of x, all vertices of $C_0 = V - (S + C)$ are adjacent to all vertices of S. Consider the case where S has at least one nonprobe. Then C_0 has only probes. By table look-up we can check if there is an embedding for the block (x, C).

Consider the case where S has only probes. If C_0 is not an independent set, it has at least one probe, say x. Again, we can check by table look-up if there is an embedding for (x, C). The algorithm in Section 4 checks if there is an embedding for $S + C_0$. If C_0 is an independent set of nonprobes, we can proceed as described prior to this theorem. □

Remark 2. There are $O(nm)$ different blocks in a graph. A rough estimate shows that each block can be processed in $O(n^2)$ time. This shows a time complexity of $O(n^3 m)$ time as a rough upperbound. Notice that for the processing of a block only vertices are considered that are at distance at most 2 from the 'handle.' This shows that the algorithm can be implemented to run in $O(n^3)$ time. However, this needs a careful analysis and this is beyond the scope of today's abstract.

6 Concluding Remarks

Many probe graph classes have been investigated recently. However, many questions remain. To mention just one: Can probe perfect graphs be recognized in polynomial time?

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