



Note

Consequences of an algorithm for bridged graphs

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Abstract

Chepoi showed that every breadth first search of a bridged graph produces a cop-win ordering of the graph. We note here that Chepoi's proof gives a simple proof of the theorem that G is bridged if and only if G is cop-win and has no induced cycle of length four or five, and that this characterization together with Chepoi's proof reduces the time complexity of bridged graph recognition. Specifically, we show that bridged graph recognition is equivalent to (C_4, C_5) -free graph recognition, and reduce the best known time complexity from $O(n^4)$ to $O(n^{3.376})$.

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1. Introduction

A *bridge* of a cycle is a path between two vertices on the cycle which is shorter than either of the paths between the vertices around the cycle. G is a *bridged graph* if every cycle of length greater than three has a bridge.

Since a chord of a cycle is exactly a bridge of length 1, every chordal graph is bridged. Bridged graphs are not hereditary, since a wheel with more than five external vertices is bridged. Bridged graphs are studied in [1,2,4–7,9]; many of these papers generalize properties of chordal graphs to the larger class. The best stated time for recognition of bridged graphs is $O(n^4)$ [6].

Let $N[v]$ denote the set consisting of v and all of its neighbors. In every nontrivial bridged graph, there is a pair of adjacent vertices x, y such that $N[x] \subseteq N[y]$ (see [1]);

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note that for any nonisolated simplicial vertex x , any neighbor y of x satisfies this condition. This generalization of simpliciality can lead to a generalization of a perfect elimination scheme as follows:

G is *cop-win* if there is an ordering v_1, v_2, \dots, v_n of the vertices such that for each $i < n$ there is a $j > i$ such that $N[v_i] \subseteq N[v_j]$ in the graph induced by v_i, \dots, v_n .

Recently, Chepoi [2] showed that cop-win orderings of bridged graphs can be constructed easily; the reversal of any breadth first ordering of a bridged graph is always a cop-win ordering.

We will show that Chepoi's theorem leads to an improved time bound for recognition of bridged graphs, using a theorem first proved in [1] that G is bridged if and only if G is cop-win and G contains no C_4 or C_5 as an induced subgraph; here, C_ℓ denotes the cycle of length ℓ . We first show that this characterization, which was previously derived as a corollary of a more difficult theorem, follows directly from Chepoi's result. We then use the characterization to reduce the time complexity of bridged graph recognition from $O(n^4)$ to $O(n^{x+1})$ where n^x is the time complexity of matrix multiplication, and show that bridged graph recognition and (C_4, C_5) -free graph recognition can be reduced to each other using linear time and space.

2. Direct proof of bridged graph characterization

We give a simplified proof of the fact that G is bridged if and only if G is cop-win and G contains no chordless C_4 and C_5 . Chepoi's result gives an easy proof of one direction (a more complicated proof is previously given in [1]); every bridged graph is cop-win, and clearly cannot have an induced C_4 or C_5 , since these cycles cannot be bridged without a chord.

Theorem 1. *A graph G is bridged if and only if G is cop-win and has no chordless cycle of length 4 or 5.*

Proof (Sufficiency). A set S of vertices is convex if S includes every shortest path with endvertices in S . For every integer $k \geq 0$, let $N^k[v]$ denote the set consisting of all vertices that can be reached from v by a path of length at most k . We will use the following known and easy to see fact on bridged graphs (cf. [4,6,7,10]):

Fact 1. *For every k and every vertex x in a bridged graph X , $N^k[x]$ is convex in X .*

Let G be a (C_4, C_5) -free cop-win graph, and consider a cop-win ordering v_1, v_2, \dots, v_n of G . By induction, $G - v_1$ is a bridged graph. Assume that G is not bridged. Then there must be a chordless, bridgeless cycle C in G , with $|C| \geq 6$ and $v_1 \in C$. Let $v_j (j > 1)$ be a vertex with $N[v_1] \subseteq N[v_j]$. As C has no chord, $v_j \notin C$. Then $C' = C - v_1 + v_j$ is a cycle in $G - v_1$ of length ≥ 6 . Since $G - v_1$ is bridged, C' has a bridge B . If $v_j \notin B$ then clearly B is also a bridge of C , a contradiction. Thus, v_j is an endpoint of B . Let $x \in C'$ be the other endpoint of B , and let a, b be the two neighbors of v_1 on C . Write $C[x, a]$, $C[x, b], \dots$, for the subpaths on C between and including the vertices x and a ,

resp., x and b . W.l.o.g. let

$$|E(C[x, a])| \geq |E(C[x, b])|.$$

As B is a bridge of C' , we have

$$|E(B)| \leq |E(C[x, b])|.$$

Since $B + v_j v_1$ is not a bridge of C , we therefore have

$$|E(B)| = |E(C[x, b])| =: k.$$

Note that $k \geq 2$. Otherwise, $|E(C[x, a])| \geq 3$ because $|C| \geq 6$, hence $B + v_j a = xv_j a$ would be a bridge of C .

In particular, b and v_j belong to $N^k[x]$. We distinguish two cases.

Case 1: $|E(C[x, a])| > k$.

Let $y \in C[x, a]$ with $|E(C[x, y])| = k$. Thus, $y \in N^k[x]$. Note that $a \notin N^k[x]$ (otherwise, a shortest x, a -path in $G - v_1$ would include a bridge of C), and ay must be an edge of C (otherwise, $B + v_j a$ would be a bridge of C). Now, yav_j is a chordless path in $G - v_1$ with $a \notin N^k[x]$. By Fact 1, v_j and y must be adjacent. But then $bv_j y$ is a bridge of C because $k \geq 2$. This contradiction settles Case 1.

Case 2: $|E(C[x, a])| = k$.

Then $a, b, v_j \in N^k[x]$. Let $y \neq v_1$ be the other neighbor of a on C , and let $z \neq v_1$ be the other neighbor of b on C . Then $y, z \in N^{k-1}[x]$. Note that v_j is nonadjacent to y and z (otherwise $yv_j b$ or $zv_j a$ would be a bridge of C as $k \geq 2$). Hence, as C has no chord, $Q = yav_j bz$ is a chordless path in $G - v_1$. By Fact 1, a shortest y, z -path P in $G - v_1$ must have length $< |E(Q)| = 4$. But then P includes a bridge of C if $k \geq 3$.

Suppose $k = 2$. Then $C = v_1 a y x z b v_1$. Let $B = xv_j$ for some vertex v . Now it is easy to see (by considering the possible neighbors of v on C and noting that C has no bridge) that G then has an induced C_4 or C_5 . This contradiction settles Case 2.

Thus G must be a bridged graph. \square

3. Recognition of bridged graphs

This section shows that the characterization of the previous section can be used to recognize bridged graphs in $O(n^{2+1})$ time, where n^α is the time needed to multiply two n by n matrices. We first reduce bridged graph recognition to (C_4, C_5) -free graph recognition, and show that the latter problem can be solved in $O(n^{2+1})$ time. Using the best current bounds [3] this gives an $O(n^{3.376})$ time bound for bridged graph recognition.

To recognize bridged graphs, we first perform breadth first search; this takes linear time, and produces (using simply the reverse of the breadth first ordering) a cop-win ordering if G is bridged. In fact, Chepoi shows [2] that if G is bridged, each vertex in the ordering is “dominated” by its parent in the breadth first search tree. It is easy to check whether the ordering is cop-win with each vertex dominated by its parent in the breadth first search tree in linear time. The algorithm for checking is very similar to algorithms for verifying that a vertex ordering is a perfect elimination scheme (see,

for example, [8]); we add neighbors of x which occur after x in the ordering to a list associated with the parent of x in the breadth first search tree, and make sure that each vertex is adjacent to every element on its list. Thus, we can reduce bridged graph recognition to (C_4, C_5) -graph recognition in linear time. We now show how to test whether an arbitrary graph G has an induced C_4 or C_5 in $O(n^{3.376})$ time.

For each vertex v , construct the graph G_v as follows. Create two vertices x_1, x_4 for each neighbor x of v , and two vertices y_2, y_3 for each nonneighbor y of v (with the exception of v itself). Add an edge between b_i and c_j if $i < j$, $i \neq 1$ or $j \neq 4$, and b and c are adjacent in G .

It should be clear that v is in an induced 4-cycle if and only if there is a path of length 2 from some b_1 to c_4 in G_v such that b and c are nonadjacent in G . Similarly, if v is in an induced 5-cycle, there will be a path of length 3 from some b_1 to c_4 such that b and c are nonadjacent in G . If there is a path of length 3 from some b_1 to c_4 with b and c nonadjacent in G , then v, b and c are part of either an induced 4-cycle or an induced 5-cycle. Thus, v is part of an induced cycle of length 4 or 5 if and only if there is a path of length 2 or 3 between some b_1 and c_4 in G_v such that b and c are nonadjacent in G .

It is well known that if A is the adjacency matrix of a graph, then $A^i[j, k]$ gives the number of paths of length i between j and k . Thus, if we are given the square and cube of the adjacency matrix for G_v , we can test whether v is part of an induced C_4 or C_5 in G in $O(n^2)$ time. Thus, the bottleneck for determining whether G has an induced C_4 or C_5 is the time to compute $O(n)$ matrix multiplications of matrices of dimension $O(n)$ by $O(n)$, giving a bound of $O(n^{3.376})$.

We have shown that the time for bridged graph recognition is $O(n + m)$ plus the time for recognizing graphs with no induced C_4 and C_5 . It is also easy to reduce the (C_4, C_5) -free graph recognition problem to bridged graph recognition, since G has no induced C_4 and C_5 if and only if G plus a universal vertex (i.e. a vertex adjacent to all other vertices) is bridged. Thus, bridged graph recognition is essentially equivalent to recognizing graphs without induced C_4 and C_5 , and both can be recognized in $O(n^{3.376})$ time.

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