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Bisplit graphs

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Abstract

An undirected graph $G=(V,E)$ is a *bisplit graph* if its vertex set can be partitioned into a stable set and a complete bipartite graph. We provide an $\mathcal{O}(|V||E|)$ time recognition algorithm for these graphs and characterize them in terms of forbidden induced subgraphs. We also discuss the problem of recognizing whether G has a stable set S such that the connected components of $G[V\setminus S]$ are more than one complete bipartite subgraph.

Keywords: Bipartite graphs; Split graphs; Comparability graphs; Induced subgraph characterization

1. Introduction

An undirected graph $G=(V,E)$ is a *bisplit graph* if its vertex set can be partitioned into three stable sets X, Y and Z such that $Y \cup Z$ induces a complete bipartite subgraph (a *bi-clique*) in G. Such a partition of V is called *proper*. We denote the class of all bisplit graphs by Φ.

In a sense, the class of bisplit graphs resembles that of split graphs, i.e., graphs whose vertex set can be partitioned into a stable set and a clique [9]. Split graphs have many interesting properties such as simple induced subgraph characterization, linear time recognition as well as polynomial time algorithms for some NP-hard problems and thus are of primary

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importance in graph theory and for graph algorithms. Replacement of "clique" in the definition of split graphs with "bi-clique", however, changes a lot, still preserving some attractive properties.

Obviously, bisplit graphs form an intermediate class between 2-colorable (i.e., bipartite) and 3-colorable graphs. It is well-known that bipartite graphs can be recognized in linear time whereas it is NP-complete to recognize whether a graph is 3-colorable [12,15]. In the present paper we refine the complexity gap between these two extremes by considering the following hierarchy of classes: A graph G is *k-bisplit* if it has a stable set X such that the induced subgraph $G[V\setminus X]$ has at most k connected components, and all these are bi-cliques. G is a *weak bisplit graph* if it is k-bisplit for some k. Note that 1-bisplit graphs are exactly the bisplit graphs.

We prove that recognizing weak bisplit graphs is NP-complete while k-bisplit graphs can be recognized in polynomial time for every fixed k. For $k = 1$, a polynomial time recognition is also given in [8]. In the present paper, we study this problem in more detail and give an $\mathcal{O}(|V||E|)$ time recognition which, to the best of our knowledge, is the fastest known recognition for these graphs.

Another observation is that bisplit graphs are transitively orientable and thus comparability graphs: If (X, Y, Z) is a proper partition of the bisplit graph G then orient all edges from X to Y, from Z to X and from Z to Y. Thus, bisplit graphs represent another poorly explored area between bipartite and comparability graphs which both have characterizations in terms of forbidden induced subgraphs: It is well-known that a graph is bipartite if and only if it contains no odd cycles [14], and for comparability graphs, a forbidden subgraph characterization was given by Gallai in [10]. In the present paper we give a forbidden induced subgraph characterization for the class of bisplit graphs in terms of minimal non-bisplit graphs that are transitively orientable.

Note that there are other problems in graph theory requiring the identification of a stable set S in a graph $G = (V, E)$ such that the subgraph induced by $V\setminus S$ has a certain property Π. Examples of such problems appeared in [3] including the case when Π is a split graph, a threshold graph, a trivially perfect graph or a tree. Also, this problem is shown in [13] to be NP-complete when Π means the property of being a cograph.

Another problem of this type is the *stable cutset problem* asking whether the input graph $G = (V, E)$ has a stable set S such that $V\setminus S$ induces a disconnected subgraph in G; this problem is NP-complete and is discussed in [2,4–6,17].

Throughout this paper, all graphs are finite, loopless and undirected. For a subset of vertices U of a graph $G = (V, E)$, we denote by $G[U]$ the subgraph of G induced by U. A vertex $v \in V$ is said to be a *neighbor* of a vertex $u \in V$ if it is adjacent to u, and a *non-neighbor* otherwise. The set of all neighbors of a vertex u is denoted $N(u)$. As usual, K_n is the complete graph on n vertices. The complete edge set between disjoint vertex sets U and W is called *join* and denoted by $U \otimes W$.

2. Recognition algorithm

In the recognition algorithm below, the first phase of a two-phase "transitive orientation" procedure from [16] is used as a subroutine. Given an arbitrary graph G, the first phase of this procedure, called *orientation phase*, assigns orientations to all its edges, the orientation being transitive if and only if G is a comparability graph. The second phase, called *verification phase,* checks whether the obtained orientation is transitive. For graphs with n vertices and m edges, the orientation phase can be implemented in $O(n + m)$ time [16], and the verification phase has the same time bound as matrix multiplication. Our algorithm only needs the orientation phase.

In an oriented graph, a vertex v is called a *source* if every edge incident to v is outgoing, and a *sink* if every edge incident to v is incoming. In the algorithm, we use the obvious fact that in a bipartite graph with any transitive orientation of the edges, every non-isolated vertex is either a source or a sink. Obviously, the following graphs are not bisplit graphs:

- K_4 (since it is not 3-colorable);
- $K_3 + K_2$ (since in a bisplit graph, the set of vertices non-adjacent to a triangle is stable);
- chordless cycles of odd length greater than 3 (since they are not transitively orientable).

Therefore, if a graph G contains no triangles, then G is bisplit if and only if it is bipartite. Assume now that G contains a triangle T with vertices a, b, c. We say that a vertex $v \notin \{a, b, c\}$ is a k-vertex for T, $k \in \{0, 1, 2, 3\}$, if v is adjacent to exactly k vertices of T. Thus, in a bisplit graph, no triangle T has a 3-vertex. Let O denotes the set of 0-vertices for T; A, B, C denote the sets of 1-vertices adjacent, respectively, to a, b or c; AB, AC, BC denote the sets of 2-vertices adjacent, respectively, to a and b, a and c, or b and c.

Our recognition algorithm is based on the following propositions, the first of which is obvious.

Proposition 1. *If G is a bisplit graph and abc is a triangle in G, then*

(i) *AB, AC, BC and O are stable sets, and $G[A]$, $G[B]$ and $G[C]$ are bipartite graphs.*
(ii) *Suppose that (X, Y, Z) is a proper partition of G. Then $O \subset X$. Moreover, if $a \in X$, $b \in Y$, $c \in Z$, then $BC \subset X$, $AC \subset Y$, and $AB \subset Z$.*

Since bisplit graphs are transitively orientable, a triangle T must have a transitive orientation. Note that in every transitive orientation of a triangle, one vertex has indegree 2 and outdegree 0, one vertex has indegree 0 and outdegree 2, and the third vertex has indegree 1 and outdegree 1. Let us show that it is correct to assign the vertex of indegree 1 and outdegree 1 in T to X.

Proposition 2. *Let $G = (V, E)$ be a bisplit graph and let $\overrightarrow{G}$ be any transitive orientation of G. Suppose that abc is a triangle in G such that $\overrightarrow{ab}, \overrightarrow{ca}, \overrightarrow{cb} \in \overrightarrow{G}$. Then $A = \emptyset$ and G admits a proper partition (X, Y, Z) with $a \in X$.*

Proof. The fact that $A = \emptyset$ follows by transitivity. Let X', Y', Z' be an arbitrary proper partition of G with $a \notin X'$; without loss of generality let $a \in Y'$, $b \in X'$, $c \in Z'$. Note that, as $\overrightarrow{ca} \in \overrightarrow{G}$, all edges of the join $Y' \otimes Z'$ are oriented from Z' to Y'. Let M be the set of all vertices in X' which have a non-neighbor in Z'. If $M = \emptyset$, the statement follows by setting $X := Y'$, $Y := X'$, $Z := Z'$. Therefore, let us assume $M \neq \emptyset$.

For each $v \in M$, let $H(v)$ be the connected component of the bipartite graph $G[X' \cup Y']$ containing v. Set $H := \bigcup_{v \in M} H(v)$. Then

$$a \notin H,$$

because if $a \in H(v)$ for some $v \in M$, then clearly $b \in H(v)$. Therefore, as $\overrightarrow{ab} \in \overrightarrow{G}$, all edges of $H(v)$ are oriented from Y' to X' by transitivity. Consider an edge (y, v) of $H(v)$ for some $y \in Y'$ (possibly $y = a$), and let $z \in Z'$ be a non-neighbor of v. Then $\overrightarrow{zy}, \overrightarrow{yv} \in \overrightarrow{G}$, but z and v are non-adjacent. This contradiction proves that $a \notin H$.

Let us set now

$$X := (Y' \backslash H) \cup (H \cap X'), \quad Y := (X' \backslash H) \cup (H \cap Y'), \quad Z := Z'.$$

By the definition of H, the sets X and Y are stable and $a \in X$. Moreover, since $M \subset X$ and $Y' \otimes Z'$, we have $Y \otimes Z$. Thus, (X, Y, Z) is a proper partition of G with $a \in X$. □

Let $\overrightarrow{G}$ be any transitive orientation of G. For a subset $U \subseteq V(G)$, let

U^0 be the set of isolated vertices in $G[U]$;
U^- be the set of sources in $\overrightarrow{G}[U]$;
U^+ be the set of sinks in $\overrightarrow{G}[U]$.

Note that if $G[U]$ is bipartite, then $U = U^0 \cup U^- \cup U^+$.

Proposition 3. *Let G be a bisplit graph and let $\overrightarrow{G}$ be any transitive orientation of G. Suppose that abc is a triangle in G such that $\overrightarrow{ab}, \overrightarrow{ca}, \overrightarrow{cb} \in \overrightarrow{G}$. Then G admits a proper partition (X, Y, Z) such that*

$$\{a\} \cup BC \cup O \cup B^+ \cup C^- \subseteq X, \quad \{b\} \cup AC \cup C^+ \subseteq Y, \quad \text{and}$$
$$\{c\} \cup AB \cup B^- \subseteq Z.$$

Proof. By Proposition 2, G admits a proper partition (X, Y, Z) with $a \in X$. Without loss of generality assume that $b \in Y$, $c \in Z$. By Proposition 1,

$$BC \cup O \subset X, \quad AC \subset Y, \quad AB \subset Z.$$

Moreover, $B \subseteq X \cup Z$ and $C \subseteq X \cup Y$. If $B^+ \cap Z = \emptyset$ and $C^- \cap Y = \emptyset$, then (X, Y, Z) is a desired proper partition. For other cases we first show:

Claim 1. *If $B^+ \cap Z \neq \emptyset$, then $C^- \subset X$. If $C^- \cap Y \neq \emptyset$, then $B^+ \subset X$.*

Proof of Claim 1. Let $u \in B^+ \cap Z$, and let $v \in B^- \cap X$ be a neighbor of u. Consider an arbitrary edge (s, t) of $G[C]$ with $s \in X$, $t \in Y$. Since $\overrightarrow{vu}, \overrightarrow{ut} \in \overrightarrow{G}$, it follows by transitivity that v, t are adjacent and $\overrightarrow{vt} \in \overrightarrow{G}$. As X is stable, $\overrightarrow{st} \in \overrightarrow{G}$, showing that $C^- \subset X$. The second part of Claim 1 can be proven similarly. □

Claim 2. $(B^- \cap X) \otimes Y$ *and* $(C^+ \cap X) \otimes Z$.

Proof of Claim 2. Let $u \in B^- \cap X$, and let $v \in B^+ \cap Z$ be a neighbor of u. As $\overrightarrow{uv} \in \overrightarrow{G}$ and all edges of the join $Y \otimes Z$ are oriented from Z to Y, u is adjacent to all vertices in Y by transitivity. The second part can be proven similarly. □

Now, if $B^+ \cap Z \neq \emptyset$, then replace X by $X \backslash (B^- \cap X) \cup (B^+ \cap Z)$, and Z by $Z \backslash (B^+ \cap Z) \cup (B^- \cap X)$. By Claims 1 and 2, (X, Y, Z) is a desired proper partition of G. The case $C^- \cap Y \neq \emptyset$ is similar. □

The strategy of the recognition algorithm is to try an assignment of vertices to classes X, Y, Z with some obvious initial assignments according to Propositions 1–3. Whenever there is a conflict of a vertex not yet assigned to vertices already assigned, we can define a new assignment or show that G is not a bisplit graph.

Algorithm 1 (*Recognition of bisplit graphs*).
Input: A graph $G = (V, E)$.
Output: "YES" if $G \in \Phi$; "NO" otherwise.

(1) Take an arbitrary vertex $v \in V$ and partition V into subsets

$$V_0 = \{v\}, V_1, V_2, \ldots, V_p,$$

where V_j denotes the subset of vertices at distance j from v [*BFS hanging from vertex* v].
(2) If for each $j \in \{1, \ldots, p\}$, V_j is a stable set, then output "YES" and STOP [*G is bipartite*]; otherwise let j be the smallest index for which V_j contains an edge e.
(3) If the endpoints of e do not have any common neighbor in V_{j-1}, then output "NO" and STOP [*G contains an induced cycle of odd length greater than three*]; otherwise let T be a triangle formed by the endpoints of e together with a common neighbor in V_{j-1}.
(4) Carry out the orientation phase of procedure "Transitive orientation" for G. If the orientation of the triangle T is not transitive, then output "NO" and STOP [*G is not a comparability graph*]; otherwise denote by a the vertex of indegree 1 and outdegree 1 in T, denote by b the vertex of indegree 2 and outdegree 0 in T and denote by c the vertex of indegree 0 and outdegree 2. Thus, T has the orientation $a \to b$, $c \to a$, $c \to b$.
(5) If $A \neq \emptyset$, then output "NO" and STOP [*G is not a comparability graph*]. If T has a 3-vertex or if one of the sets O, AB, AC, BC is not stable, then output "NO" and STOP [*G is not a bisplit graph by Proposition 3*]. If $G[B]$ or $G[C]$ is not bipartite, then output "NO" and STOP [*G is not a bisplit graph*]. Else determine the sets B^0, B^-, B^+, C^0, C^-, C^+ given by the orientation phase. Let
$X_0 := \{a\} \cup BC \cup O \cup B^+ \cup C^-$,
$Y_0 := \{b\} \cup AC \cup C^+$,
$Z_0 := \{c\} \cup AB \cup B^-$.

(6) (a) determine all B^0- and C^0-vertices conflicting with X_0, Y_0, Z_0:
$B^1_X :=$ the vertices of B^0 that have a neighbor in Z_0 or a non-neighbor in Y_0;
$B^1_Z :=$ the vertices of B^0 that have a neighbor in X_0;

C_X^1 := the vertices of C^0 that have a neighbor in Y_0 or a non-neighbor in Z_0;
C_Y^1 := the vertices of C^0 that have a neighbor in X_0;

(b) If $B_X^1 \cap B_Z^1 \neq \emptyset$ or $C_X^1 \cap C_Y^1 \neq \emptyset$, then output "NO" and STOP [*a vertex has a double conflict*]

(c) $X_1 := X_0 \cup B_X^1 \cup C_X^1$, $Y_1 := Y_0 \cup C_Y^1$, $Z_1 := Z_0 \cup B_Z^1$.

(d) $B^1 := B^0 \backslash (B_X^1 \cup B_Z^1)$, $C^1 := C^0 \backslash (C_X^1 \cup C_Y^1)$[*the remaining vertices to be assigned*]
If $B^1 = B^0$ and $C^1 = C^0$, then GOTO (8).

(7) $k := 1$;
repeat
$k := k + 1$;

(a) [*determine new conflicting vertices of the next round*]
B_X^k := the vertices of B^{k-1} that have a non-neighbor in C_Y^{k-1} [B_X^k *vertices have to be assigned to X*]
B_Z^k := the vertices of B^{k-1} that have a neighbor in C_X^{k-1} [B_Z^k *vertices have to be assigned to Z*]
C_X^k := the vertices of C^{k-1} that have a non-neighbor in B_Z^{k-1} [C_X^k *vertices have to be assigned to X*]
C_Y^k := the vertices of C^{k-1} that have a neighbor in B_X^{k-1} [C_Y^k *vertices have to be assigned to Y*]

(b) If $B_X^k \cap B_Z^k \neq \emptyset$ or $C_X^k \cap C_Y^k \neq \emptyset$, then output "NO" and STOP [*a vertex has a double conflict*]

(c) $X_k := X_{k-1} \cup B_X^k \cup C_X^k$, $Y_k := Y_{k-1} \cup C_Y^k$, $Z_k := Z_{k-1} \cup B_Z^k$.

(d) $B^k := B^{k-1} \backslash (B_X^k \cup B_Z^k)$, $C^k := C^{k-1} \backslash (C_X^k \cup C_Y^k)$ [*the remaining vertices to be assigned*]

until $B_X^k \cup B_Z^k \cup C_X^k \cup C_Y^k = \emptyset$.

(8) $X := X_k \cup B^k$, $Y := Y_k \cup C^k$, $Z := Z_k$.

(9) If (X, Y, Z) is a proper partition of G, then output "YES", otherwise "NO".

Theorem 4. *Algorithm* 1 *correctly recognizes bisplit graphs with n vertices and m edges in time* $\mathrm{O}(nm)$.

Proof. *Correctness*. We have to show that G is a bisplit graph if the algorithm says "YES", and G is not a bisplit graph if the algorithm says "NO". Since the only steps where the algorithm says "YES" are step (2) (bipartite graphs are obviously bisplit graphs) and step (9) where the algorithm checks once more whether the obtained partition is proper, the first claim is fulfilled. The first answer "NO" can be given in step (3) (where an induced odd cycle is found) which is correct, then in step (4) (where G is not a comparability graph). The answer "NO" in step (5) is correct by Propositions 1 and 2. In case $\overrightarrow{G}$ is not transitive, the answer "NO" in steps (6) and (7) is clearly correct; in the other case, the correctness follows from Proposition 3 (conflicts assign vertices to classes X, Y, Z and double conflicts show that G is not a bisplit graph). Finally, it is easy to see that, after the repeat-loop is finished, if (X_k, Y_k, Z_k) is a proper partition of the subgraph $G_k = G[X_k \cup Y_k \cup Z_k]$, then the partition (X, Y, Z) in Step (8) is proper for the entire graph G.

Time bound. With an appropriate data structure, a single execution of the repeat loop can be implemented in time $O(n+m)$. Since the algorithm carries out Step (7) at most n times, the total time complexity of this step is $O(nm)$. This also bounds the time complexity of any other step of the algorithm. □

3. Forbidden induced subgraph characterization

In view of the discussion in the introduction, every incomparability graph is forbidden in Φ. In this section, we characterize bisplit graphs by a set H of minimal forbidden induced subgraphs that are comparability graphs. In the set H we distinguish two infinite graph families $H^0 = \{H_1^0, H_2^0, H_3^0, \ldots\}$ and $H^1 = \{H_1^1, H_2^1, H_3^1, \ldots\}$, defined by induction, as follows.

Induction basis. H_1^0 is defined to be a K_4 on the vertices x_0, y_0, x_1, y_1, while H_1^1 is defined to be a $K_3 + K_2$, i.e., the disjoint union of a K_3 on the vertices x_0, y_0, z_0 and a K_2 on the vertices x_1, y_1.

Induction rules.

1. For an even k and $i \in \{0, 1\}$, the graph H_k^i is obtained from H_{k-1}^i by

- deleting the edge (x_{k-1}, y_{k-1});
- adding a pair of non-adjacent vertices x_k, y_k;
- connecting x_k to vertex y_{k-1} and to each y_j with even $j < k-1$, and connecting y_k to vertex x_{k-1} and to each x_j with even $j < k-1$.

2. For an odd k and $i \in \{0, 1\}$, the graph H_k^i is obtained from H_{k-1}^i by

- connecting vertices x_{k-1} and y_{k-1} by an edge;
- adding a pair of adjacent vertices x_k, y_k;
- connecting x_k to each vertex y_j with even $j < k-1$, and connecting y_k to each vertex x_j with even $j < k-1$.

In the sequel, in order to distinguish induced subgraphs isomorphic to H_k^0 or H_k^1 with $k > 1$, we shall refer to these graphs as

$$H_k^0 \begin{bmatrix} x_0, x_1, x_2, x_3, \ldots, x_{k-1}, x_k \\ y_0, y_1, y_2, y_3, \ldots, y_{k-1}, y_k \end{bmatrix}$$

and

$$H_k^1(z_0) \begin{bmatrix} x_0, x_1, x_2, x_3, \ldots, x_{k-1}, x_k \\ y_0, y_1, y_2, y_3, \ldots, y_{k-1}, y_k \end{bmatrix},$$

respectively. In addition, we shall use the fact that any graph of the form H_k^1 can be uniquely characterized by its degree sequence, which ranges from 1 to $k+1$ both for the x_j- and y_j-vertices (Fig. 1).

Theorem 5. *A comparability graph $G = (V, E)$ is bisplit if and only if it does not contain any graph in H as an induced subgraph.*

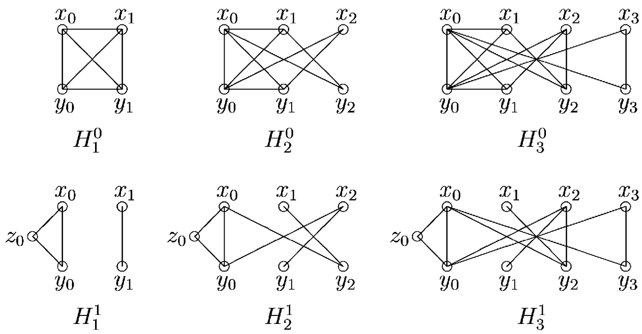

Fig. 1. The first six graphs in the family *H*.

Proof. *Necessity*. Let G be a bisplit graph with a proper partition (X, Y, Z). Assume by contradiction that G contains an induced subgraph $H_k^i \in H$ with some $i \in \{0, 1\}$ and $k > 0$. Denote $W_j = \{x_j, y_j\}$, $j = 0, 1, \ldots, k$. We use induction on j to show that

(a) for every odd j, $W_j \subseteq X$,
(b) for every even j, $W_j \subseteq Y \cup Z$; furthermore, $x_j \in Y$ and $y_j \in Z$, or vice versa.

Suppose first that $i = 0$. Clearly $H_1^0 \notin \Phi$ and hence $k > 1$. Consequently, x_1 is not adjacent to y_1. Assuming that x_0 belongs to X and y_0 to Z, we conclude that $x_1, y_1 \in Y$, $x_2 \in X$, $y_2 \in Z$. But now the partition (X, Y, Z) is no longer proper, since $y_1 \in Y$ is not adjacent to $y_2 \in Z$. The only way to avoid this contradiction is to assume that, up to symmetry, $x_0 \in Y$ and $y_0 \in Z$, and hence $x_1, y_1 \in X$ and $x_2 \in Y$, $y_2 \in Z$.

Suppose now that $i = 1$. No vertex in W_1 has a neighbor in the triangle x_0, y_0, z_0. Therefore, each vertex of W_1 belongs to X, since otherwise $G[Y \cup Z]$ is not a bi-clique. Consequently, $(x_1, y_1) \notin E$ and hence $k > 1$. By the definition of H_k^1, x_2 is adjacent to y_1, and y_2 is adjacent to x_1. Thus, $W_2 \subseteq Y \cup Z$. If both vertices of W_2 belong to Y, then x_0 does not belong to Z, since x_0 is not adjacent to x_2. Analogously, y_0 and z_0 do not belong to Z. This contradicts the fact that exactly one vertex of the triangle x_0, y_0, z_0 must belong to Z. A similar contradiction is obtained if both vertices of W_2 belong to Z. Therefore, one of the vertices in W_2 belongs to Y, and another one to Z. Without loss of generality we assume that $x_2 \in Y$ and $y_2 \in Z$, implying that $z_0 \in X$, $x_0 \in Y$ and $y_0 \in Z$. Thus we have established the induction basis for (a) and (b).

To make the inductive step, let $j > 2$. Since x_j is adjacent to y_0, the vertex x_j does not belong to Z. By analogy, y_j does not belong to Y. For odd j, we know that x_j is not adjacent to y_{j-1}, but by the induction hypothesis y_{j-1} is in Z, and hence x_j is not in Y. Thus x_j belongs to X, and similarly $y_j \in X$. For even j, the vertex x_j is adjacent to y_{j-1}, and by the induction hypothesis y_{j-1} is in X, hence x_j is not in X. Consequently, $x_j \in Y$, and by analogy $y_j \in Z$. This completes the proof of (a) and (b).

Let k be odd. Then, on the one hand, the vertices x_k and y_k are adjacent due to the construction of H_k^i. On the other hand, $W_k \subseteq X$ by (a), a contradiction. If k is even, then x_k is not adjacent to y_k in H_k^i, but x_k is adjacent to y_k by (b), again a contradiction. The necessity is proved.

Sufficiency. Let G be a comparability graph containing no graph in H as an induced subgraph. Our goal is to show that G belongs to Φ. The proposition is trivial if G does not contain any triangle. Indeed, in that case G is bipartite, since no comparability graph contains an odd cycle of length greater than 3 as an induced subgraph.

From now on, we assume that G contains a triangle $T = (a, b, c)$. As in the recognition algorithm, we suppose that T admits the following transitive orientation: $\overrightarrow{ab}, \overrightarrow{ca}, \overrightarrow{cb}$. We keep also all the notations introduced in the algorithm. In addition, we denote $G_k := G[X_k \cup Y_k \cup Z_k]$ for $k \geqslant 0$.

It is a simple exercise to verify that if (X_k, Y_k, Z_k) is a proper partition of the graph G_k for each $k \geqslant 0$, then so is the partition (X, Y, Z) of the graph G defined in Step (8) of the algorithm. The fact that (X_k, Y_k, Z_k) is a proper partition of G_k will be proved by induction on k according to the following plan.

The basis:

Claim 6 will show that the subsets X_0, Y_0, Z_0 form a proper partition of the subgraph G_0;

Claim 7 will state that $B_X^1 \cap B_Z^1 = \emptyset$ and $C_X^1 \cap C_Y^1 = \emptyset$, proving thereby the correctness of the definition of G_1;

Claim 8 will prove that the subsets X_1, Y_1, Z_1 form a proper partition of the subgraph G_1.

The induction step $(k > 1)$: assuming that $G_j \in \Phi$ for each $j < k$,

Claim 9 will assert that $B_X^k \cap B_Z^k = \emptyset$ and $C_X^k \cap C_Y^k = \emptyset$, justifying the construction of G_k;

Claim 11 will conclude that $G_k \in \Phi$.

Claim 6. *The subsets X_0, Y_0, Z_0 form a proper partition of the subgraph G_0.*

Proof. To prove the claim, we have to show that X_0, Y_0, Z_0 are stable sets, and each vertex in Y_0 is adjacent to each vertex in Z_0.

First, observe that the subsets B^+, B^-, C^+, C^- are stable, as can be seen directly from their definition. Also, AB, AC, BC are obviously stable, otherwise G would contain a K_4. Taking into account the transitivity condition, we conclude that no vertex in BC is adjacent to a vertex in O, and every vertex in AB is adjacent to every vertex in AC. The rest can be proved as follows.

X_0 is a stable set. No vertex x in $BC \cup O$ is adjacent to a vertex y in B^+. Indeed, if x would be adjacent to y, then any neighbor $z \in B^-$ of y would be adjacent to x by transitivity, implying either a $K_4 = G[x, y, z, b]$ (if $x \in BC$) or a $K_3 + K_2 = G[x, y, z, a, c]$ (if $x \in O$). Analogously, no vertex in $BC \cup O$ is adjacent to a vertex in C^-.

No vertex x in B^+ is adjacent to a vertex y in C^-. To prove this, denote by z_1 a vertex in B^- adjacent to x, and by z_2 a vertex in C^+ adjacent to y. The existence of the edge (x, y) would imply by transitivity the existence of the edges (z_1, y), (x, z_2) and (z_1, z_2), and therefore the existence of a $K_4 = G[x, y, z_1, z_2]$.

Y_0 is a stable set. No vertex x in AC is adjacent to a vertex y in C^+, since otherwise any neighbor $z \in C^-$ of y would be adjacent to x by transitivity, implying the existence of a $K_4 = G[x, y, z, c]$. Similarly, Z_0 is a stable set.

Each vertex $x \in Y_0$ *is adjacent to each vertex* $y \in Z_0$. Assume that y belongs to B^-, and let $z \in B^+$ denote a neighbor of y. By transitivity, if y is not adjacent to x, then z also is not adjacent to x. Consequently, $x \in C^+$, else G contains an induced $K_3+K_2=G[x, a, c, y, z]$. By transitivity, no neighbor $t \in C^-$ of x is adjacent to y or to z. But now the vertices x, c, t, y, z induce a $K_3 + K_2$ in G. This contradiction proves that x is adjacent to $y \in B^-$. The remaining case follows by symmetry. □

Claim 7. $B_X^1 \cap B_Z^1 = \emptyset$ *and* $C_X^1 \cap C_Y^1 = \emptyset$.

Proof. Due to symmetry we restrict ourselves to the proof of the fact that $B_X^1 \cap B_Z^1 = \emptyset$. Assume by contradiction that a vertex u belongs both to B_X^1 and to B_Z^1. By the definition of B_Z^1, u has a neighbor $v \in X_0 = \{a\} \cup BC \cup O \cup C^- \cup B^+$. By the definition of B_X^1, there is a vertex w, which is either a neighbor of u in $Z_0 = \{c\} \cup AB \cup B^-$ or a non-neighbor of u in $Y_0 = \{b\} \cup AC \cup C^+$. Obviously $v \neq a$, $w \neq b$, $w \neq c$, $v \notin B^+$, $w \notin B^-$ (see the definition of B^0). We analyze below the remaining cases.

If u is adjacent to $v \in BC \cup O$ and to $w \in AB$, then v is adjacent to w by transitivity. But then G contains either a $K_4 = G[b, u, v, w]$ (if $v \in BC$) or

$$H_2^0 \begin{bmatrix} b, a, v \\ w, u, c \end{bmatrix}$$

(if $v \in O$).

If u is adjacent to $v \in BC \cup O$ and non-adjacent to $w \in AC$, then v is not adjacent to w by transitivity. But then G contains either a $K_3 + K_2 = G[a, c, w, u, v]$ (if $v \in O$) or

$$H_2^0 \begin{bmatrix} c, a, u \\ b, v, w \end{bmatrix}$$

(if $v \in BC$).

If u is adjacent to $v \in BC \cup O$ and non-adjacent to $w \in C^+$, denote by t a neighbor of w in C^-. By transitivity, v is not adjacent to w and hence is not adjacent to t. For the same reason, u is not adjacent to t. But then G contains a $K_3 + K_2$ induced either by vertices b, u, v, w, t (if $v \in BC$) or by vertices c, w, t, u, v (if $v \in O$).

If u is adjacent to $v \in C^-$, we denote by s a neighbor of v in C^+. By transitivity, u is adjacent to s. For the same reason, if u is adjacent to $w \in AB$, then w is adjacent to v and s, which implies the existence of a $K_4=G[s, u, v, w]$. If u is not adjacent to $w \in AC \cup C^+$, then w is not adjacent to v and s. In case that $w \in AC$, G contains a $K_3 + K_2 = G[u, v, s, a, w]$. If $w \in C^+$, we denote by t a neighbor of w in C^-. From the definition of C^- it follows that t is not adjacent to v, and by transitivity t is not adjacent to u. But now G contains either a $K_3 + K_2 = G[u, v, s, w, t]$ (if $(t, s) \notin E$) or

$$H_2^0 \begin{bmatrix} c, t, u \\ s, v, w \end{bmatrix}$$

(if $(t, s) \in E$). □

Claim 8. *The subsets* X_1, Y_1, Z_1 *form a proper partition of the subgraph* G_1.

Proof. Taking into account Claims 6, 7, we have to show only that

(1) no vertex u in B^1_X has a neighbor v in C^1_X;
(2) every vertex u in B^1_Z is adjacent to every vertex v in C^1_Y.

To prove (1), we denote by s either a neighbor of u in Z_0 or a non-neighbor of u in Y_0. By t we denote either a neighbor of v in Y_0 or a non-neighbor of v in Z_0. To the contrary we assume that u is adjacent to v, and consider the following cases that exhaust, up to symmetry, all possibilities for u and v.

If u is adjacent to $s \in AB$, and v is not adjacent to $t \in AB \cup B^-$, then by transitivity s is adjacent to v, and t is not adjacent to u. As a consequence, $s \neq t$, and by Claim 6 $(s, t) \notin E$. Therefore, $t \in B^-$, since otherwise G would contain the subgraph

$$H_2^0\begin{bmatrix} b, a, v \\ s, u, t \end{bmatrix}.$$

Denote by w a neighbor of t in B^+. The vertex w is adjacent neither to v (by transitivity) nor to u (by the definition of B^0). Now if w is not adjacent to s, then G contains a $K_3 + K_2 = G[s, u, v, t, w]$, and if w is adjacent to s, then G contains

$$H_2^0\begin{bmatrix} b, w, v \\ s, u, t \end{bmatrix}.$$

Let u be non-adjacent to $s \in AC$, and v non-adjacent to $t \in AB \cup B^-$. By Claim 6 $(s, t) \in E$, and by transitivity $(t, u), (s, v) \notin E$. Therefore, $t \in B^-$, else G contains a $K_3 + K_2 = G[a, s, t, u, v]$. Denote by w a neighbor of t in B^+. Just as above, w is adjacent neither to v nor to u. But now if w is not adjacent to s, then the vertices a, c, s, t, u, v, w induce in G a H_2^1, and if w is adjacent to s, then G contains a $K_3 + K_2 = G[s, t, w, u, v]$.

Let u be adjacent to $s \in AB$, and v adjacent to $t \in AC$. By Claim 6 $(s, t) \in E$, and by transitivity $(t, u), (s, v) \in E$. But now $G[s, t, u, v]$ is a K_4.

If u is not adjacent to $s \in C^+$, and v is not adjacent to $t \in B^-$, we denote by p a neighbor of t in B^+, and by q a neighbor of s in C^-. Then $(u, q), (v, p) \notin E$ by transitivity, $(u, t), (u, p), (v, s), (v, q) \notin E$ by definition, and $(s, t) \in E$ by Claim 6. Now either the vertices p, q, s, t contain a triangle inducing together with u, v a $K_3 + K_2$, or G contains the subgraph

$$H_2^1(p)\begin{bmatrix} t, q, u \\ b, v, s \end{bmatrix}.$$

To prove (2), we denote by s a neighbor of u in X_0, and by t a neighbor of v in X_0. By contradiction we assume that u is not adjacent to v. This implies that u is not adjacent to t, since otherwise $t \in BC \cup O$, in which case the graph is not transitively orientable. Similarly, v is not adjacent to s. Therefore, $s \neq t$, and hence $(s, t) \notin E$ by Claim 6. Up to symmetry we have to analyze the following cases.

If $s \in BC$ and $t \in BC$, then G contains the subgraph

$$H_2^0\begin{bmatrix} b, s, v \\ c, t, u \end{bmatrix}.$$

If $s, t \in O$, then G contains

$$H_2^1(a)\begin{bmatrix} b, s, v \\ c, t, u \end{bmatrix}.$$

If $s \in BC$, $t \in O$, then G contains a $K_3 + K_2 = G[b, u, s, t, v]$.

If $s \in BC \cup O$ and $t \in B^+$, denote by p a neighbor of t in B^-. By transitivity p is adjacent to v, and by definition p is not adjacent to u. But now if p is not adjacent to s, then G contains a $K_3 + K_2 = G[t, p, v, u, s]$. If p is adjacent to s, then G contains either

$$H_2^0\begin{bmatrix} b, s, v \\ p, t, u \end{bmatrix}$$

(if $s \in BC$) or

$$H_2^1(t)\begin{bmatrix} p, u, c \\ v, a, s \end{bmatrix}$$

(if $s \in O$).

Finally, let $s \in C^-$ and $t \in B^+$. Denote by p a neighbor of t in B^- and by q a neighbor of s in C^+. By definition, u is not adjacent to p, and v is not adjacent to q. By transitivity u is adjacent to q, and v is adjacent to p. Consequently, p is adjacent to s, else G contains a $K_3 + K_2 = G[t, p, v, u, s]$, and similarly q is adjacent to t. This implies by transitivity that $(p, q) \in E$. But now G contains the subgraph

$$H_2^0\begin{bmatrix} q, s, v \\ p, t, u \end{bmatrix}. \qquad \square$$

Claim 8 completes establishing the basis of our induction for $k = 0, 1$, making it now possible to apply the induction step for $k > 1$. In what follows we shall denote by b_X^j a vertex in B_X^j, by b_Z^j a vertex in B_Z^j, by c_X^j a vertex in C_X^j, and by c_Y^j a vertex in C_Y^j.

From the definition of the subsets B_X^j, B_Z^j, C_X^j and C_Y^j with $j > 1$ it follows that

(a) each vertex $u \in B_X^j \cup B_Z^j$ is adjacent to

- each vertex in C_Y^i with $i < j - 2$ (otherwise u would belong to B_X^{i+1}),
- each vertex in Y_0 (otherwise u would belong to B_X^1),
- no vertex in Z_0 (otherwise u would belong to B_X^1),
- no vertex in X_0 (otherwise u would belong to B_Z^1),
- no vertex in C_X^i with $i < j - 2$ (otherwise u would belong to B_Z^{i+1});

(b) each vertex $u \in C_X^j \cup C_Y^j$ is adjacent to

- each vertex B_Z^i with $i < j - 2$ (otherwise u would belong to C_X^{i+1}),
- each vertex in Z_0 (otherwise u would belong to C_X^1),
- no vertex in Y_0 (otherwise u would belong to C_X^1),

- no vertex in X_0 (otherwise u would belong to C_Y^1),
- no vertex in B_X^i with $i < j-2$ (otherwise u would belong to C_Y^{i+1}).

By the induction hypothesis we shall assume that for each $1 < j < k$,

(c) $B_X^j \cap B_Z^j = \emptyset$ and $C_X^j \cap C_Y^j = \emptyset$ (in other words, b_Z^j is adjacent to c_Y^{j-1}, while b_X^j is not adjacent to c_X^{j-1}, and c_Y^j is adjacent to b_Z^{j-1}, while c_X^j is not adjacent to b_X^{j-1});

(d) (X_j, Y_j, Z_j) is a proper partition of the subgraph G_j, which means in particular that b_X^j is not adjacent to c_X^j, and b_Z^j is adjacent to c_Y^j.

Keeping (a)–(d) in mind, we prove the following two claims.

Claim 9. *For $k > 1$, $B_X^k \cap B_Z^k = \emptyset$ and $C_X^k \cap C_Y^k = \emptyset$.*

Proof. Due to symmetry, it is sufficient to prove that $C_X^k \cap C_Y^k = \emptyset$. By contradiction, let c^k be a vertex in $C_X^k \cap C_Y^k$. By the definition of C_X^k, c^k has a non-neighbor $b_Z^{k-1} \in B_Z^{k-1}$. Analogously, if $k-1 > 1$, then b_Z^{k-1} has a neighbor $c_X^{k-2} \in C_X^{k-2}$. Proceeding in this way, we obtain a sequence of vertices

$$\mathscr{A} = c^k, b_Z^{k-1}, c_X^{k-2}, b_Z^{k-3} \dots ,$$

where c_X^j is not adjacent to b_Z^{j-1}, and b_Z^j is adjacent to c_X^{j-1}.

Similarly, since c^k belongs to C_Y^k, we associate to it a sequence of vertices

$$\mathscr{B} = c^k, b_X^{k-1}, c_Y^{k-2}, b_X^{k-3} \dots ,$$

where c_Y^j is adjacent to b_X^{j-1}, and b_X^j is not adjacent to c_Y^{j-1}.

Case 1: *k is odd*. In that case the sequence $\mathscr{A}$ ends with $c_X^1 \in C_X^1$ and $\mathscr{B}$ with $c_Y^1 \in C_Y^1$. The proof will be given in several steps: (1) we first establish some preliminary facts, (2) then prove a helpful lemma, and (3) finally provide an exhaustive analysis of the case.

(1) *Preliminaries*.

From the definition of C_X^1 it follows that c_X^1 has a conflict with a vertex v, which is either a neighbor of c_X^1 in AC or a non-neighbor of c_X^1 in $AB \cup B^-$. From Claim 7 we know that if v is adjacent to c_X^1, then it is not adjacent to c_Y^1, and vice versa. Similarly, by the definition of C_Y^1 the vertex c_Y^1 has a neighbor $w \in BC \cup O \cup B^+$, which is not adjacent to c_X^1 due to Claim 7. Taking into account observations (a)–(d), we conclude that only two possibilities have to be examined: $(v, w) \in E$ and $(v, w) \notin E$. In step 3 of the proof, we analyze these two options for each possible arrangement of v and w.

(2) *Helpful lemma*.

Lemma 10. *If $u \in B^-$ is a neighbor of the vertex $w \in B^+$, then u is adjacent to every vertex of the form c_X^j or c_Y^j. If $u \in B^+$ is a neighbor of the vertex $v \in B^-$, then u has no neighbors of the form c_X^j or c_Y^j.*

Proof. Consider the two following ordered sequences of vertices:

$$\mathscr{C} = \{c_Y^1, c_Y^3, c_Y^5, \ldots, c_Y^{k-4}, c_Y^{k-2}, c^k, c_X^{k-2}, c_X^{k-4}, \ldots, c_X^5, c_X^3, c_X^1\},$$

$$\mathscr{D} = \{b_X^2, b_X^4, b_X^6, \ldots, b_X^{k-3}, b_X^{k-1}, b_Z^{k-1}, b_Z^{k-3}, \ldots, b_Z^6, b_Z^4, b_Z^2\}.$$

We shall denote by $\mathscr{C}(i)$ and $\mathscr{D}(i)$ the ith elements of the respective sequences.

If $u \in B^-$ is a neighbor of $w \in B^+$, then u is adjacent to $\mathscr{C}(1) = c_Y^1$ by transitivity. Assume now that u is adjacent to the vertices $\mathscr{C}(1), \mathscr{C}(2), \ldots, \mathscr{C}(i-1)$. If, to the contrary, u is not adjacent to $\mathscr{C}(i)$, then the vertices

$$w, u, \mathscr{C}(1), \mathscr{D}(1), \mathscr{C}(2), \mathscr{D}(2), \ldots, \mathscr{D}(i-1), \mathscr{C}(i)$$

induce the subgraph H_{i-1}^1 in G. To be more specific, we indicate that

- w represents the vertex z_0 of H_{i-1}^1,
- $\mathscr{C}(1), \mathscr{C}(2), \ldots, \mathscr{C}(i)$ represent the x_j-vertices of H_{i-1}^1 with $\mathscr{C}(j)$ having degree $i-j+1$ for $j = 1, \ldots, i$,
- $u, \mathscr{D}(1), \mathscr{D}(2), \ldots, \mathscr{D}(i-1)$ represent the y_j-vertices of H_{i-1}^1 with $\mathscr{D}(j)$ having degree j for $j = 1, \ldots, i-1$, and u having degree i.

Thus, the first part of the lemma is proven by induction. The proof of the second part is analogous. Indeed, if $u \in B^+$ is a neighbor of $v \in B^-$, then u is not adjacent to $\mathscr{C}(k) = c_X^1$ by transitivity. Suppose u is not adjacent to the vertices $\mathscr{C}(k), \mathscr{C}(k-1), \ldots, \mathscr{C}(k-i+1)$. Then the assumption that u is adjacent to $\mathscr{C}(k-i)$ would lead to the contradiction that the vertices $u, v, \mathscr{C}(k), \mathscr{D}(k), \mathscr{C}(k-1), \mathscr{D}(k-1), \ldots, \mathscr{C}(k-i+1), \mathscr{D}(k-i+1), \mathscr{C}(k-i)$ induce the subgraph H_i^1 in G. □

(3) *Analysis*.

Case 1.1: $(v, w) \in E$.

If $v \in AB \cup B^-$ and $w \in BC \cup O \cup B^+$, then G contains the forbidden subgraph

$$H_{k-1}^1(w) \begin{bmatrix} v, b_X^2, b_Z^2, b_X^4, b_Z^4, \ldots, b_X^{k-3}, b_Z^{k-3}, b_X^{k-1}, b_Z^{k-1} \\ c_Y^1, c_X^1, c_Y^3, c_X^3, c_Y^5, \ldots, c_X^{k-4}, c_Y^{k-2}, c_X^{k-2}, c^k \end{bmatrix}.$$

If $v \in AC$ and $w \in O$, then the following subgraph is forbidden:

$$H_{k+1}^0 \begin{bmatrix} c, a, b_Z^2, w, b_Z^4, b_X^2, b_Z^6, \ldots, b_X^{k-5}, b_Z^{k-1}, b_X^{k-3}, b_X^{k-1} \\ v, c_X^1, b, c_X^3, c_Y^1, c_X^5, c_Y^3, \ldots, c_X^{k-2}, c_Y^{k-4}, c^k, c_Y^{k-2} \end{bmatrix}.$$

If $v \in AC$ and $w \in BC \cup B^+$, then the following forbidden subgraph arises:

$$H_k^0 \begin{bmatrix} x, w, b_Z^2, b_X^2, b_Z^4, b_X^4, \ldots, b_Z^{k-3}, b_X^{k-3}, b_Z^{k-1}, b_X^{k-1} \\ v, c_X^1, c_Y^1, c_X^3, c_Y^3, c_X^5, \ldots, c_Y^{k-4}, c_X^{k-2}, c_Y^{k-2}, c^k \end{bmatrix},$$

where x denotes either the vertex c if $w \in BC$, or a vertex $u \in B^-$ adjacent to w if $w \in B^+$ (u is adjacent to v by Claim 6, and u is adjacent to every vertex of the form c_X^j or c_Y^j by Lemma 10).

Case 1.2: $(v, w) \notin E$.

If $v \in AB \cup B^-$ and $w \in O$, then G contains the forbidden subgraph

$$H_k^1(x)\begin{bmatrix} v, w, b_Z^2, b_X^2, b_Z^4, \dots, b_Z^{k-3}, b_X^{k-3}, b_Z^{k-1}, b_X^{k-1} \\ b, c_X^1, c_Y^1, c_X^3, c_Y^3, \dots, c_Y^{k-4}, c_X^{k-2}, c_Y^{k-2}, c^k \end{bmatrix},$$

where x denotes either the vertex a if $v \in AB$, or a vertex $u \in B^+$ adjacent to v if $v \in B^-$ (u is not adjacent to w by Claim 6 and has no neighbors of the form c_X^j or c_Y^j by Lemma 10).

If $v \in AB \cup B^-$ and $w \in B^+$, or $v \in B^-$ and $w \in BC$, then the following forbidden subgraph appears:

$$H_k^1(w)\begin{bmatrix} x, b_X^2, v, b_X^4, b_Z^2, b_X^6, b_Z^4, \dots, b_Z^{k-5}, b_X^{k-1}, b_Z^{k-3}, b_Z^{k-1} \\ c_Y^1, y, c_Y^3, c_X^1, c_Y^5, c_X^3, c_Y^7, \dots, c_Y^{k-2}, c_X^{k-4}, c^k, c_X^{k-2} \end{bmatrix},$$

where

- x is a vertex in B^- adjacent to w, and y stands for a, when $v \in AB$, $w \in B^+$;
- x is a vertex in B^- adjacent to w, and y is a vertex in B^+ adjacent to v, when $v \in B^-$, $w \in B^+$;
- x stands for c, and y is a vertex in B^+ adjacent to v, when $v \in B^-$, $w \in BC$.

If $v \in AC$ and $w \in BC \cup O \cup B^+$, then G contains the forbidden subgraph

$$H_{k-1}^1(c_X^1)\begin{bmatrix} b_Z^2, w, b_Z^4, b_X^2, b_Z^6, b_X^4, \dots, b_X^{k-5}, b_Z^{k-1}, b_X^{k-3}, b_X^{k-1} \\ v, c_X^3, c_Y^1, c_X^5, c_Y^3, c_X^7, \dots, c_X^{k-2}, c_Y^{k-4}, c^k, c_Y^{k-2} \end{bmatrix}.$$

If $v \in AB$ and $w \in BC$, then the following subgraph is forbidden:

$$H_{k+1}^0\begin{bmatrix} c, w, v, b_X^2, b_Z^2, b_X^4, b_Z^4, \dots, b_X^{k-3}, b_Z^{k-3}, b_X^{k-1}, b_Z^{k-1} \\ b, a, c_Y^1, c_X^1, c_Y^3, c_X^3, c_Y^5, \dots, c_X^{k-4}, c_Y^{k-2}, c_X^{k-2}, c^k \end{bmatrix}.$$

Case 2: *k is even*. In that case the sequence $\mathscr{A}$ ends with $b_Z^1 \in B_Z^1$ and $\mathscr{B}$ with $b_x^1 \in B_X^1$. The proof of this case follows exactly the same plan as the proof of case 1. Moreover, with some care, the text of the proof of case 2 can be obtained from that of case 1 by exchanging "*AB*" with "*AC*", "*c*" with "*b*" ("*c*" and "*b*" are vertices), "C_X^j" with "B_X^j, "C_Y^j" with "B_Z^j", "c_X^j" with "b_X^j", "c_Y^j" with "b_Z^j", "C^+" with "B^-", and "C^-" with "B^+".

Taking into account the parity change, the sequences $\mathscr{C}$ and $\mathscr{D}$ in Lemma 10, as well as the forbidden subgraphs in the analysis part, must be rewritten more carefully. For instance, $\mathscr{C}$ should read as

$$\mathscr{C} = \{b_Z^1, b_Z^3, b_Z^5, \dots, b_Z^{k-3}, b_Z^{k-1}, b_X^{k-1}, b_X^{k-3}, \dots, b_X^5, b_X^3, b_X^1\}.$$

The descriptions of forbidden subgraphs can be adapted to the case of even k as follows. We distinguish two parts within the brackets: before the dots and after the dots. In the first part, we exchange row 1 with row 2, and then "*c*" with "*b*", "c_X^j" with "b_X^j", and "c_Y^j" with "b_Z^j". In the second part, we only change the subscripts: in the first row exchange "*X*" with "*Z*", and in the second row "*X*" with "*Y*". □

Claim 11. *For $k > 1$, the subsets X_k, Y_k, Z_k form a proper partition of the subgraph of G_k.*

Proof. Taking into account (a)–(d) and Claim 9, we have to show only that

(1) no vertex in B_X^k is adjacent to a vertex in C_X^k;
(2) each vertex in B_Z^k is adjacent to each vertex in C_Y^k.

To prove (1), assume to the contrary that a vertex $b_X^k \in B_X^k$ is adjacent to a vertex $c_X^k \in C_X^k$. We associate with b_X^k a sequence of vertices

$$\mathscr{A} = b_X^k, c_Y^{k-1}, b_X^{k-2}, c_Y^{k-3} \ldots,$$

where b_X^j is not adjacent to c_Y^{j-1}, and c_Y^j is adjacent to b_X^{j-1}. Analogously, with the vertex c_X^k we associate a sequence of vertices

$$\mathscr{B} = c_X^k, b_Z^{k-1}, c_X^{k-2}, b_Z^{k-3} \ldots,$$

where c_X^j is not adjacent to b_Z^{j-1}, and b_Z^j is adjacent to c_X^{j-1}.

Case 1: *k is odd*. In this case the sequence $\mathscr{A}$ ends with $b_X^1 \in B_X^1$ and $\mathscr{B}$ with $c_X^1 \in C_X^1$. From the definition of B_X^1 it follows that b_X^1 has a conflict with a vertex v, which is either a neighbor of b_X^1 in AB or a non-neighbor of b_X^1 in $AC \cup C^+$. Similarly, c_X^1 has either a neighbor $w \in AC$ or a non-neighbor $w \in AB \cup B^-$ by the definition of C_X^1. In analysis of possibilities for v and w, we use the following lemma, which can be proven by analogy with Lemma 10. □

Lemma 12. *If $u \in C^-$ is a neighbor of the vertex $v \in C^+$, then u has no neighbors of the form b_X^j or b_Z^j. If $u \in B^+$ is a neighbor of the vertex $w \in B^-$, then u has no neighbors of the form c_X^j or c_Y^j.*

In addition, we introduce the following notations:
$F(x)$ stands for

$$H_{k-1}^1(b_X^1)\begin{bmatrix} x, b_X^3, b_Z^2, b_X^5, b_Z^4, \ldots, b_X^{k-2}, b_Z^{k-3}, b_X^k, b_Z^{k-1} \\ c_Y^2, c_X^1, c_Y^4, c_X^3, c_Y^6, \ldots, c_X^{k-4}, c_Y^{k-1}, c_X^{k-2}, c_X^k \end{bmatrix},$$

$G(x)$ stands for

$$H_{k+1}^0\begin{bmatrix} v, b_X^1, w, b_X^3, b_Z^2, b_X^5, b_Z^4, \ldots, b_X^{k-2}, b_Z^{k-3}, b_X^k, b_Z^{k-1} \\ b, x, c_Y^2, c_X^1, c_Y^4, c_X^3, c_Y^6 \ldots, c_X^{k-4}, c_Y^{k-1}, c_X^{k-2}, c_X^k \end{bmatrix},$$

and $H(x)$ stands for

$$H_k^1(x)\begin{bmatrix} w, b_X^1, b_Z^2, b_X^3, b_Z^4, \ldots, b_Z^{k-3}, b_X^{k-2}, b_Z^{k-1}, b_X^k \\ v, c_X^1, c_Y^2, c_X^3, c_Y^4 \ldots, c_Y^{k-3}, c_X^{k-2}, c_Y^{k-1}, c_X^k \end{bmatrix}.$$

Case 1.1: $v \in AB$. Then v is adjacent to c_X^1, otherwise G contains $F(v)$ as an induced subgraph.

If $w \in AC$, then w is adjacent to v (Claim 6) and hence to b_X^1, since otherwise G would contain $F(w)$. But then G contains

$$H_k^0 \begin{bmatrix} v, b_X^1, b_Z^2, b_X^3, b_Z^4, b_X^5, \ldots, b_Z^{k-3}, b_X^{k-2}, b_Z^{k-1}, f b_X^k \\ w, c_X^1, c_Y^2, c_X^3, c_Y^4, c_X^5 \ldots, c_Y^{k-3}, c_X^{k-2}, c_Y^{k-1}, c_X^k \end{bmatrix}.$$

If $w \in AB \cup B^-$, then w is not adjacent to v (Claim 6) and is not adjacent to b_X^1, since otherwise G would contain $F(w)$. We hence conclude that $w \in B^-$, else G contains $G(a)$. Let u be a vertex in B^+ adjacent to w. By Lemma 12, u has no neighbors of the form c_X^j or c_Y^j. If v is adjacent to u, then G contains $G(u)$. If v is not adjacent to u, then the following subgraph is forbidden:

$$H_k^1(b_X^1) \begin{bmatrix} v, b_X^3, w, b_X^5, b_Z^2, b_X^7, b_Z^4, \ldots, b_Z^{k-5}, b_X^k, b_Z^{k-3}, b_Z^{k-1} \\ c_Y^2, u, c_Y^4, c_X^1, c_Y^6, c_X^3, c_Y^8 \ldots, c_Y^{k-1}, c_X^{k-4}, c_X^k, c_X^{k-2} \end{bmatrix}.$$

Case 1.2: $v \in AC \cup C^+$ and $w \in B^-$. By Claim 6, v and w are adjacent. By the definition of B^0, w is not adjacent to $b_X^1 \in B^0$. Denote by u a vertex in B^+ adjacent to w. Due to Lemma 12 u has no neighbors of the form c_X^j or c_Y^j.

In case that $v \in AC$, we have $(v, c_X^1) \notin E$, since otherwise $F(v)$ arises. As a result, if $(v, u) \in E$ then $H(u)$ appears, and if $(v, u) \notin E$ we have the following forbidden subgraph:

$$H_{k+1}^1(a) \begin{bmatrix} c, b_X^1, w, b_X^3, b_Z^2, b_X^5, b_Z^4, \ldots, b_X^{k-2}, b_Z^{k-3}, b_X^k, b_Z^{k-1} \\ v, u, c_Y^2, c_X^1, c_Y^4, c_X^3, c_Y^6 \ldots, c_X^{k-4}, c_Y^{k-1}, c_X^{k-2}, c_X^k \end{bmatrix}.$$

If $v \in C^+$, then v is not adjacent to $c_X^1 \in C^0$ by the definition of C^0. Let t be a vertex in C^- adjacent to v. By Lemma 12, t has no neighbors of the form b_X^j or b_Z^j, and by Claim 6, t is not adjacent to u. Next, if t is adjacent to w, then G contains $H(t)$, and if u is adjacent to v, then G contains $H(u)$. If both $(t, w) \notin E$ and $(u, v) \notin E$, then the following subgraph is forbidden:

$$H_{k+1}^1(u) \begin{bmatrix} w, t, b_Z^2, b_X^1, b_Z^4, b_X^3, \ldots, b_X^{k-4}, b_Z^{k-1}, b_X^{k-2}, b_X^k \\ b, c_X^1, v, c_X^3, c_Y^2, c_X^5, \ldots, c_X^{k-2}, c_Y^{k-3}, c_X^k, c_Y^{k-1} \end{bmatrix}.$$

The remaining cases follow by symmetry.

Case 2: *k is even*. In this case the sequence $\mathscr{A}$ ends with $c_Y^1 \in C_Y^1$ and $\mathscr{B}$ with $b_Z^1 \in B_Z^1$. From the definition of C_Y^1 it follows that c_Y^1 has a neighbor v in $BC \cup O \cup B^+$. Similarly, b_Z^1 has a neighbor $w \in BC \cup O \cup C^-$ by the definition of B_Z^1. The vertices v and w belong to X_0 and hence are non-adjacent by Claim 6. Moreover, the vertex v is not adjacent to b_Z^1, since otherwise G would contain the induced subgraph

$$H_{k-1}^1(v) \begin{bmatrix} b_Z^1, b_X^2, b_Z^3, b_X^4, \ldots, b_Z^{k-3}, b_X^{k-2}, b_Z^{k-1}, b_X^k \\ c_Y^1, c_X^2, c_Y^3, c_X^4 \ldots, c_Y^{k-3}, c_X^{k-2}, c_Y^{k-1}, c_X^k \end{bmatrix}.$$

Symmetrically, w is not adjacent to c_Y^1. The following lemma is an analog of Lemmas 10 and 12.

Lemma 13. *If $u \in B^-$ is a neighbor of the vertex $v \in B^+$, then u is adjacent to every vertex of the form c_X^j or c_Y^j. If $u \in C^+$ is a neighbor of the vertex $w \in C^-$, then u is adjacent to every vertex of the form b_X^j or b_Z^j.*

Also, we write $F(x, y)$ to denote the graph

$$H^0_{k+1}\begin{bmatrix} x, v, b^1_Z, b^2_X, b^3_Z, b^4_X, \dots, b^{k-3}_Z, b^{k-2}_X, b^{k-1}_Z, b^k_X \\ y, w, c^1_Y, c^2_X, c^3_Y, c^4_X \dots, c^{k-3}_Y, c^{k-2}_X, c^{k-1}_Y, c^k_X \end{bmatrix},$$

and $G(x)$ to denote the graph

$$H^1_k(v)\begin{bmatrix} x, b^2_X, b^1_Z, b^4_X, b^3_Z, b^6_X, \dots, b^{k-2}_X, b^{k-3}_Z, b^k_X, b^{k-1}_Z \\ c^1_Y, w, c^3_Y, c^2_X, c^5_Y, c^4_X, \dots, c^{k-4}_X, c^{k-1}_Y, c^{k-2}_X, c^k_X \end{bmatrix}.$$

If $v \in BC$ and $w \in BC$, then G contains the induced subgraph $F(c, b)$. If $v \in BC$ and $w \in O$, then G contains the induced subgraph $G(c)$. If $v \in O$ and $w \in O$, then G contains the induced subgraph

$$H^1_{k+1}(a)\begin{bmatrix} c, v, b^1_Z, b^2_X, b^3_Z, b^4_X, \dots, b^{k-3}_Z, b^{k-2}_X, b^{k-1}_Z, b^k_X \\ b, w, c^1_Y, c^2_X, c^3_Y, c^4_X \dots, c^{k-3}_Y, c^{k-2}_X, c^{k-1}_Y, c^k_X \end{bmatrix}.$$

Now let $v \in B^+$ and u be a vertex in B^- adjacent to v. If $w \in BC \cup O$, then G contains the induced subgraph $G(u)$. If $w \in C^-$, we consider a vertex $t \in C^+$ adjacent to w. By Claim 6, u and t are adjacent. Moreover, t is adjacent to v, since otherwise G would contain the induced subgraph

$$H^1_k(w)\begin{bmatrix} b^1_Z, v, b^3_Z, b^2_X, b^5_Z, \dots, b^{k-4}_X, b^{k-1}_Z, b^{k-2}_X, b^k_X \\ t, c^2_X, c^1_Y, c^4_X, c^3_Y, \dots, c^{k-2}_X, c^{k-3}_Y, c^k_X, c^{k-1}_Y \end{bmatrix}.$$

Symmetrically, u is adjacent to w. But then G contains the induced subgraph $F(u, t)$.

The remaining cases follow by symmetry.

The proof of (2) is similar to the proof of (1) and hence is omitted. □

We have completed the induction showing that the subsets X_k, Y_k, Z_k form a proper partition of the subgraph G_k for each $k \geqslant 0$. Therefore, as mentioned before, (X, Y, Z) is a proper partition of G. The theorem is proved. □

4. Generalizations and extensions

Recall that a graph G is k-bisplit if it has a stable set X such that the induced subgraph $G[V\setminus X]$ has at most k connected components, and all these are bi-cliques. Note that the set of all isolated vertices in $G[V\setminus X]$ is considered as a single bi-clique. Recall also that a graph G is a weak bisplit graph if it is k-bisplit for some k. Note that weak bisplit graphs are still 3-colorable but no longer comparability graphs (the chordless cycles of odd length at least five are 2-bisplit graphs). In contrast to bisplit graphs, it will be shown below that recognizing weak bisplit graphs is NP-complete. Moreover, we will show that recognizing k-bisplit graphs can be done in polynomial time for every fixed k.

For both results we need variants of the SATISFIABILITY problem:

2SAT: Let $\mathscr{C}$ be a collection of clauses over a set of Boolean variables, each of which contains exactly two literals. Is there a truth assignment satisfying $\mathscr{C}$?

It is well-known that 2SAT can be solved in linear time [1,7].

1-IN-3 3SAT: Let $\mathscr{C}$ be a collection of clauses over a set of Boolean variables, each of which contains exactly three unnegated literals, i.e., variables. Is there a truth assignment satisfying $\mathscr{C}$ such that each clause has exactly one true variable?

This variant of the problem is known to be NP-complete [11].

Theorem 14. *For every fixed $k \geqslant 1$, k-bisplit graphs are recognizable in polynomial time.*

Proof. To develop a polynomial time recognition of k-bisplit graphs, we need to introduce some more terminology. By analogy with bisplit graphs, we call a partition $V = X \cup W$ *proper* if X is a stable set such that the subgraph H induced by W has at most k connected components each of which is a bi-clique. Let us call any maximal subset of vertices of H with the same neighborhood (in H) a *canonical class*. Clearly, any canonical class is a stable set, any two canonical classes are disjoint and H has a unique partition into canonical classes. By picking an arbitrary vertex in each canonical class of H we obtain an induced subgraph which is called the *characteristic graph* of H. Alternatively, the characteristic graph can be obtained by contracting each canonical class into a single vertex.

With this terminology we can say that a graph $G = (V, E)$ is k-bisplit if and only if G admits a partition $V = X \cup W$ into a stable set X and a graph $H = G[W]$ such that the characteristic graph of H has at most $2k$ vertices each of which is of degree 1, except possibly for a single vertex which is isolated in H. If M is a subset of W such that $H[M]$ is the characteristic graph of H, we shall say that G admits a proper partition *with respect to* the subgraph induced by M.

To solve the recognition problem, we

(1) generate all subsets $M \subseteq V(G)$ with at most $2k$ vertices,
(2) check whether each vertex of the graph $G[M]$ is of degree 1, except possibly for a single vertex which is isolated, and if so
(3) determine whether G admits a proper partition with respect to $G[M]$.

The first two tasks in this list are obviously polynomially solvable. Below we show that question (3) also can be answered in polynomial time.

Assume that G admits a proper partition $V = X \cup W$ with respect to $G[M]$ and let $W_1, \dots, W_p$ be the canonical classes of $G[W]$. Denote by v_j the vertex of M that belongs to W_j, and let $A = A_M$ be the adjacency matrix of $G[M]$, i.e., $A(i, j) = 1$ if v_i is adjacent to v_j, and $A(i, j) = 0$ otherwise. In particular, $A(i, i) = 0$ for any $i = 1, \dots, p$. Then clearly a vertex $u \in W$ belongs to W_j if and only if $N(u) \cap M = N(v_j) \cap M$. This suggests the following approach to question (3).

Given a subset $M \subseteq V$, we classify the vertices of $V - M$ into subsets $U_0, U_1, \dots, U_p$ so that $U_j := \{u \in V - M \mid N(u) \cap M = N(v_j) \cap M\}$ for $j > 0$ and $U_0 = V - M - \bigcup_{j>0} U_j$. In other words, for $j > 0$, U_j is the set of candidate vertices for inclusion in the canonical class containing v_j, and U_0 is the set of vertices that must go to X in any proper partition of G with respect to $G[M]$. Therefore, up to this point, $X = U_0$ and $W = M$ and the canonical classes of $G[M]$ are defined by $W_j = \{v_j\}$, $j = 1, \ldots, p$. Obviously if U_0 is not a stable set, then the procedure can be terminated at this point. Otherwise, for each $j > 0$ we determine whether the vertices of U_j can be assigned to W_j or to X so that the partition $X \cup W$ remains proper. To this end, we first define the assignment enforced by the current partition, i.e., as long as possible we do the following:

- if a vertex $u \in U_j$ has a neighbor in X, we assign u to W_j;
- if $u \in U_j$ has a neighbor $v \in W_i$ with $A(j, i) = 0$ we assign u to X;
- if $u \in U_j$ has a non-neighbor $v \in W_i$ with $A(j, i) = 1$we assign u to X.

If this sequence of assignments results in a partition $X \cup W$ which is not proper, we conclude that G has no proper partition with respect to the given subgraph $G[M]$. If the partition $X \cup W$ is proper, then the rest can be done by a reduction to the 2SAT problem as follows.

With each vertex $u \in \bigcup_{j>0} U_j$ we associate a Boolean variable x_u. With pairs of vertices $u \in U_i$ and $v \in U_j$ we associate clauses of two literals in the following way.

If u is adjacent to v and $A(i, j) = 1$, then we create a clause $\overline{x}_u \vee \overline{x}_v$. If u is adjacent to v and $A(i, j) = 0$, we create two clauses $\overline{x}_u \vee \overline{x}_v$ and $x_u \vee x_v$. If u is not adjacent to v and $A(i, j) = 1$, we create a clause $x_u \vee x_v$. If u is not adjacent to v and $A(i, j) = 0$, no clause is created.

Now it is a simple exercise to verify that the 2SAT formula consisting of the created clauses is satisfiable if and only if G admits a proper partition with respect to $G[M]$. In particular, if the formula is satisfiable, then a proper partition of G is obtained by assigning each vertex u with $x_u = true$ to X and each vertex u with $x_u = false$ to W. □

Theorem 15. *Recognizing weak bisplit graphs is NP-complete, even for 3-colorable comparability graphs.*

Proof. Clearly, the problem belongs to NP. To prove the NP-completeness, we will use a reduction from 1-In-3 3Sat without negative literals.

Let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be a collection of m clauses with variable set $U = \{v_1, v_2, \ldots, v_n\}$ such that every clause C_i contains exactly three variables, $C_i = \{c_{i1}, c_{i2}, c_{i3}\}$, where each literal c_{ij} $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant 3)$ is a v_k for some suitable k. We shall construct a 3-colorable comparability graph $G = G(\mathscr{C})$ such that G is weak bisplit if and only if $\mathscr{C} \in$ 1-In-3 3Sat.

For each variable $v_k \in U$, let $G(k)$ be a triangle with a labeled vertex v_k. For each clause $C_i = \{c_{i1}, c_{i2}, c_{i3}\}$, let $G(C_i)$ be the graph shown in Fig. 2.

The following properties of $G(C_i)$ can be proved easily by inspection.

Proposition 16. *$G(C_i)$ is weak bisplit. Every stable set X of $G(C_i)$ with the property that each connected component of $G(C_i) - X$ is a bi-clique must contain exactly two of the labeled vertices c_{i1}, c_{i2}, c_{i3}, and every two vertices of c_{ij} $(1 \leqslant j \leqslant 3)$ can be extended to such a stable set X of $G(C_i)$.*

Let us construct the graph G taking as its vertex set the union of the vertex sets of all graphs $G(k)$ and all graphs $G(C_i)$. The edge set of G will include all the edges of the graphs

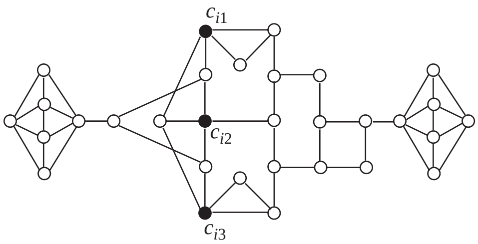

Fig. 2. The graph $G(C_i)$.

$G(k)$ and $G(C_i)$ and the following additional edges: if v_k is the variable c_{ij} in clause C_i, then the vertex v_k of the triangle $G(k)$ is connected to the vertex c_{ij} of $G(C_i)$ by an edge. Let us show that G is weak bisplit if and only if $\mathscr{C} \in$ 1-In-3 3Sat.

First, suppose that G is weak bisplit, and let X be a stable set of G such that each connected component of $G - X$ is a bi-clique. We claim that for every k and every i,

$$v_k \in X \iff \text{the neighbor } c_{ij} \text{ of } v_k \text{ in } G(C_i) \text{ does not belong to } X. \qquad (*)$$

The direction "⇒" is clear because X is a stable set. For the other direction, assume that $c_{ij} \notin X$. Note that by construction, c_{ij} has a neighbor in $G(C_i)$ that is not in X, and v_k has a neighbor in the triangle $G(k)$ that is also not in X. Therefore, v_k must belong to X, since otherwise there would be an induced path P_4 outside X, contradicting the fact that $G - X$ consists of disjoint bi-cliques. The direction "⇐" follows.

Now, assign the value true to the variable v_k if $v_k \in X$; otherwise assign to v_k the value false. Since in every $G(C_i)$ exactly one of c_{i1}, c_{i2}, c_{i3} is outside X (Proposition 16), $(*)$ shows that every clause C_i has exactly one true variable in the above assignment. Thus, $\mathscr{C} \in$ 1-In-3 3Sat.

Second, suppose that $\mathscr{C} \in$ 1-In-3 3Sat, and consider a truth assignment for the variables such that every C_i has exactly one true variable. Let us construct a set X_0 in the following way: in each $G(k)$ put into X_0 the vertex v_k if the variable v_k is true; otherwise put into X_0 one of two neighbors of v_k in $G(k)$. In each $G(C_i)$ put into X_0 those two vertices of c_{i1}, c_{i2}, c_{i3} whose corresponding variables have value false. By Proposition 16 the stable set X_0 can be extended to a stable set X of G such that $G - X$ consists of disjoint bi-cliques. Thus, G is weak bisplit.

Observe that each $G(k)$ admits a transitive orientation such that v_k is a sink, and each $G(C_i)$ admits a transitive orientation such that the vertices c_{i1}, c_{i2}, c_{i3} are sources. Hence G admits a transitive orientation by directing the edges $v_k c_{ij}$ from c_{ij} to v_k. Finally, G is 3-colorable because it is a K_4-free comparability graph (K_4-free perfect graphs are 3-colorable). □

References

[1] B. Aspvall, M.F. Plass, R.E. Tarjan, A linear-time algorithm for testing the truth of certain quantified Boolean formulas, Inform. Process. Lett. 8 (1979) 121–123.

[2] A. Brandstädt, F.F. Dragan, V.B. Le, T. Szymczak, On stable cutsets in graphs, Discrete Appl. Math. 105 (2000) 39–50.
[3] A. Brandstädt, V.B. Le, T. Szymczak, The complexity of some problems related to graph 3-colorability, Discrete Appl. Math. 89 (1998) 59–73.
[4] V. Chvátal, Recognizing decomposable graphs, J. Graph Theory 8 (1984) 51–53.
[5] D.G. Corneil, J. Fonlupt, Stable set bonding in perfect graphs and parity graphs, J. Combin. Theory Ser. B 59 (1993) 1–14.
[6] C.M.H. de Figueiredo, S. Klein, The NP-completeness of multi-partite cutset testing, Congr. Numer. 119 (1996) 217–222.
[7] S. Even, A. Itai, A. Shamir, On the complexity of timetable and multicommodity flow problems, SIAM J. Comput. 5 (1976) 691–703.
[8] T. Feder, P. Hell, S. Klein, R. Motwani, List partitions, SIAM J. Discrete Math. 16 (2003) 449–478.
[9] S. Földes, P.L. Hammer, Split graphs, Congr. Numer. 19 (1977) 311–315.
[10] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66.
[11] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, W.H. Freeman, San Francisco, 1979.
[12] M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, Theoret. Comput. Sci. 1 (1976) 237–267.
[13] C.T. Hoàng, V.B. Le, On P_4-transversals of perfect graphs, Discrete Math. 216 (2000) 195–210.
[14] D. König, Über Graphen und ihre Anwendungen auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916) 435–465.
[15] L. Lovász, Coverings and colorings of hypergraphs, Congr. Numer. VIII (1973) 3–12.
[16] R.M. McConnell, J.P. Spinrad, Modular decomposition and transitive orientation, Discrete Math. 201 (1999) 189–241.
[17] A. Tucker, Coloring graphs with stable cutsets, J. Combin. Theory Ser. B 34 (1983) 258–267.