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Note

On the complexity of 4-coloring graphs without long induced paths

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Abstract

We show that deciding if a graph without induced paths on nine vertices can be colored with 4 colors is an NP-complete problem, improving a previous NP-completeness result proved by Woeginger and Sgall in 2001. The complexity of 4-coloring graphs without induced paths on five vertices remains open. We show that deciding if a graph without induced paths or cycles on five vertices can be colored with 4 colors can be done in polynomial time. A step in our algorithm uses the well-known and deep fact due to Grötschel, Lovász and Schrijver stating that perfect graphs can be optimally colored in polynomial time.

Keywords: Graph coloring; P_5-free graph; Computational complexity; Induced path

1. Introduction

Given an integer $k \geq 1$, a k-coloring of a graph is an assignment of k colors $1, \ldots, k$, to the vertices of the graph such that adjacent vertices receive different colors. Given a graph G, the k-COL problem asks whether G admits a k-coloring. A path on t vertices is denoted by P_t. Graphs without induced P_t are called P_t-free. For $t \leq 3$, k-COL on P_t-free graphs is trivial. On P_4-free graphs (also called *cographs*), it is well-known (see, e.g., [4]) that k-COL is solvable in linear time. Recently, the following results have been obtained:

Theorem 1 (*[15,12]*). 3-COL *is solvable in polynomial time on P_5-free graphs.*

Indeed, 3-COL can be solved on P_5-free graphs in time $O(n^\alpha)$; see the recent survey paper [14] for more information (n and m are the vertices, respectively, edge number of the input graphs, $O(n^\alpha)$ is the time needed to perform an $n \times n$ matrix multiplication; currently, $\alpha \approx 2.376$).

Theorem 2 (*[13]*). 3-COL *is solvable in time $O(mn^\alpha)$ on P_6-free graphs.*

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Theorem 3 *([15]).* 4-Col *is NP-complete on* P_{12}*-free graphs, and* 5-Col *is NP-complete on* P_8*-free graphs.*

In Section 2 we will improve the NP-completeness result on 4-coloring stated in the first part of Theorem 3, by showing that 4-Col is NP-complete on P_9-free graphs. Our reduction is a modification from a given one in [15].

The complexity of deciding if a P_5-free graph is 4-colorable is still an open question. By considering subclasses of P_5-free graphs, the following partial results are known: The chromatic number of $(P_5, \overline{P_5})$-free graphs [6], and of $(P_5, K_{1,\Delta})$-free graphs [10] (Δ is any given constant and $K_{1,\Delta}$ is the star with Δ edges[1]) can be determined in polynomial time. Hence, 4-Col is polynomial on these subclasses of P_5-free graphs. In [8], based on the fact that every connected P_5-free graph has either a dominating clique or a dominating P_3 [1], a polynomial-time algorithm to decide if a P_5-free graph without dominating K_4 can be colored with 4 colors has been designed where K_t is the complete graph on t vertices. In particular, 4-Col is polynomial on (P_5, K_4)-free graphs. In Section 3 we will give a polynomial-time algorithm to decide if a (P_5, C_5)-free graph can be colored with 4 colors where C_t is the cycle on t vertices.

For a vertex v in a graph $G = (V, E)$, let $N(v) = \{u \mid uv \in E\}$ denote the neighborhood of v in G. For $U \subseteq V$, set $N(U) := \bigcup_{v \in U} N(v) \setminus U$, and the subgraph of G induced by U is denoted by $G[U]$. Throughout this paper, all subgraphs are understood to be induced subgraphs. If $\mathcal{F}$ is a set of graphs, a graph G is called $\mathcal{F}$-free if none of its induced subgraphs is isomorphic to a member of $\mathcal{F}$. The complement of G is denoted by $\overline{G}$. The chromatic number $\chi(G)$ is the smallest integer k such that G is k-colorable.

A graph G is called perfect if, for all induced subgraphs H of G, $\chi(H)$ equals to the maximum number of pairwise adjacent vertices in H. The strong perfect graph theorem (SPGT) [2] states that a graph is perfect if and only if it contains no C_t and $\overline{C_t}$ with odd $t \geq 5$. It is known that perfect graphs can be recognized in polynomial time [3], and that the chromatic number of perfect graphs can be determined in polynomial time [7].

Finally, for a coloring $f : V(G) \to \{1, 2, \ldots\}$ of G and $X \subseteq V(G)$ we write $f(X)$ for the set of colors by f appearing in X, that is, $f(X) = \{f(v) : v \in X\}$.

2. NP-completeness result

In this section we prove that 4-Col is NP-complete on P_9-free graphs by reducing the well-known NP-complete problem 3-SAT (cf. [5]) to 4-Col restricted on P_9-free graphs.

Given a 3-SAT formula $F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ over the set X of Boolean variables, where $C_i = (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ with $\ell_{ij} \in X \cup \overline{X}$, $1 \leq j \leq 3$, we will create a P_9-free graph $G = G(F)$ such that F is satisfied if and only if G is 4-colorable as follows.

- For each variable $x \in X$ let $G(x)$ denote the P_2 with vertices $v(x)$ and $v(\overline{x})$.
- For each clause $C_i = (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ let $G(C_i)$ be the graph depicted in Fig. 1, where the literals ℓ_{ij} of C_i are presented by the vertices c_{ij}.

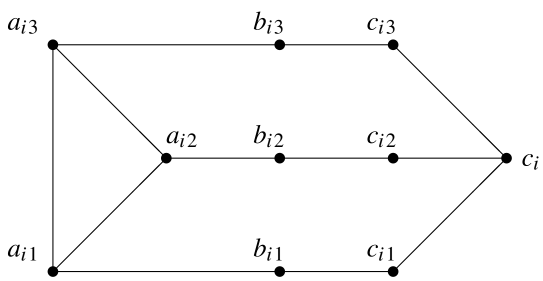

Fig. 1. The ten-vertex clause gadget $G(C_i)$.

Then the graph $G = G(F)$ is obtained from all $G(x)$, all $G(C_i)$, and the dummy vertex d by adding the following additional edges:

[1] More generally, given constants t and D, the vertex number of P_t-free graphs with diameter at most D is bounded.

- $\{c_i, v(\ell)\}$, $1 \le i \le m$, $\ell \in X \cup \overline{X}$,
- $\{b_{ij}, v(\ell)\}$, $1 \le i \le m$, $1 \le j \le 3$, and $\ell \in X \cup \overline{X}$,
- $\{d, v\}$, $v \in G(C_i)$, $1 \le i \le m$,
- $\{c_{ij}, v(\ell)\}$ for all $1 \le i \le m$, $1 \le j \le 3$, and $\ell \in X \cup \overline{X}$ with $\ell_{ij} = \ell$.

Lemma 1. *G is P_9-free.*

Proof. If u is a vertex in G, a path having u as one of its endvertices is called a u-path. We will show the following claims:

Claim 1: Every induced d-path has at most four vertices.

Claim 2: For each $\ell \in X \cup \overline{X}$, every $v(\ell)$-path not containing d has at most four vertices.

Assuming the claims, the proof of Lemma 1 goes as follows: Let P be an induced path in G. If $d \in P$, P has at most seven vertices by Claim 1. So let us assume that $d \notin P$. If P contains a vertex $v(\ell)$ for some $\ell \in X \cup \overline{X}$, P has at most seven vertices by Claim 2. Thus, we may assume further that $P \cap (\{v(\ell) : \ell \in X \cup \overline{X}\} \cup \{d\}) = \emptyset$. Then P must belong to some $G(C_i)$. Now, it is easy to verify that the ten-vertex clause gadget $G(C_i)$ has no induced paths on nine vertices. Thus, P has at most eight vertices, and Lemma 1 follows.

It should be noted that each $G(C_i)$, hence G, contains an induced P_8 on vertices $b_{i1}, c_{i1}, c_i, c_{i2}, b_{i2}, a_{i2}, a_{i3}, b_{i3}$.

In the proofs of the claims we will use the following notion: If v is a vertex of an induced u-path then v^+ denotes the neighbor of v (if any) not belonging to the subpath between u and v.

Proof of Claim 1. Let P be a maximal induced d-path with at least three vertices, and consider the index i such that $d^+ \in G(C_i)$. Then $P \cap G(C_i) = \{d^+\}$, $d^+ \in \{c_i, c_{i1}, c_{i2}, c_{i3}, b_{i1}, b_{i2}, b_{i3}\}$, and $d^{++} = v(\ell)$ for some $\ell \in X \cup \overline{X}$.

If $d^+ \in \{c_i, b_{i1}, b_{i2}, b_{i3}\}$, then $P = dd^+v(\ell)$ because d^+ is adjacent to all vertices in G_x, $x \in X$, and d is adjacent to all vertices in $G(C_{i'})$, $1 \le i' \le m$, and Claim 1 follows.

So, let $d^+ = c_{ij}$ for some $1 \le j \le 3$, and assume that P has a fourth vertex d^{+++}. Then $d^{+++} = v(\overline{\ell})$ and $P = dd^+v(\ell)v(\overline{\ell})$ as every neighbor of $v(\overline{\ell})$ different from $v(\ell)$ is adjacent to d. Claim 1 follows.

Proof of Claim 2. Fix an arbitrary vertex $v = v(\ell)$, and consider a maximal induced v-path P with at least three vertices and not containing d.

If $v^+ \in \{v(\ell') : \ell' \in X \cup \overline{X}\}$, then $v^+ = v(\overline{\ell})$ (as $v(\overline{\ell})$ is the only neighbor of $v(\ell)$ in $\{v(\ell') : \ell' \in X \cup \overline{X}\}$), and $v^{++} = c_{ij}$ for some i, j (as the neighbors c_i, b_{ij} of $v(\overline{\ell})$ are adjacent to $v(\ell)$). Now, note that c_{ij} has exactly $v^+ = v(\overline{\ell})$ as its neighbor in $\{v(\ell') : \ell' \in X \cup \overline{X}\}$) and the other neighbors c_i and b_{ij} of c_{ij} (recall that P does not contain the vertex d) are adjacent to all vertices in $\{v(\ell') : \ell' \in X \cup \overline{X}\}$), $P = vv^+v^{++}$, and Claim 2 follows.

So, let $v^+ \in G(C_i)$ for some i. Then $v^+ \ne c_{ij}$, $1 \le j \le 3$, because the neighbors c_i, b_{ij} of c_{ij} are adjacent to $v(\ell)$. We distinguish two cases.

CASE 1: $v^+ = c_i$.

If $v^{++} = v(\ell')$ for some $\ell' \ne \ell$ then $P = v(\ell)c_iv(\ell')$ or $P = v(\ell)c_iv(\ell')c_{i'j'}$ for some $i' \ne i$ and some $1 \le j' \le 3$. If $v^{++} = c_{ij}$ for some $1 \le j \le 3$, then $P = v(\ell)c_ic_{ij}$ (as the neighbor b_{ij} of $v^{++} = c_{ij}$ is adjacent to $v(\ell)$, and c_i is adjacent to all $v(\ell')$, $\ell' \in X \cup \overline{X}$.) Case 1 is settled.

CASE 2: $v^+ = b_{ij}$ for some $1 \le j \le 3$.

If $v^{++} \in G(C_i)$ then $P = v(\ell)b_{ij}c_{ij}$ or $P = v(\ell)b_{ij}a_{ij}a_{ij'}$ for some $j' \ne j$. If $v^{++} = v(\ell')$ for some $\ell' \ne \ell$ then $P = v(\ell)b_{ij}v(\ell')c_{i'j'}$ for some i', j' where $i' = i$ or $j' = j$ (but not both) is possible. Case 2 is settled, and Claim 2 follows. □

Lemma 2. *If G is 4-colorable then F is satisfiable.*

Proof. Let $f : V(G) \to \{1, 2, 3, 4\}$ be a 4-coloring of G. Then

$$\text{for all } 1 \le i \le m,\ |f(\{b_{i1}, b_{i2}, b_{i3}\})| \ge 2. \tag{1}$$

For, if $f(b_{i1}) = f(b_{i2}) = f(b_{i3}) =: c$ for some i and some $c \in \{1, 2, 3, 4\}$ then, as d is adjacent to all b_{ij} and a_{ij}, $c \ne f(d)$ and $f(a_{ij}) \in \{1, 2, 3, 4\} \setminus \{c, f(d)\}$. This is impossible due to the fact that a_{i1}, a_{i2}, a_{i3} induce a triangle in G, hence (1). Moreover,

$$|f(\{v(\ell) : \ell \in X \cup \overline{X}\}) \cup \{f(d)\}| = 2. \tag{2}$$

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To see this, observe first that, as for each $x \in X$, $v(x)$ and $v(\overline{x})$ are adjacent, $|f(\{v(\ell) : \ell \in X \cup \overline{X}\})| \geq 2$. If $|f(\{v(\ell) : \ell \in X \cup \overline{X}\}) \cup \{f(d)\}| \geq 3$ then for all $1 \leq i \leq m$, all b_{i1}, b_{i2}, b_{i3} are colored with the same color (as they are adjacent to d and to all $v(\ell)$, $\ell \in X \cup \overline{X}$). This is impossible due to (1), hence (2).

Let, without loss of generality, $f(d) = 1$. Then, by (2), we may assume that

$$f(\{v(\ell) : \ell \in X \cup \overline{X}\}) = \{1, 2\}.$$

Now, a truth assignment σ for the variables $x \in X$ can be defined as follows:

$$\sigma(x) := \texttt{true} \text{ if } f(v(x)) = 1; \text{ otherwise, } \sigma(x) := \texttt{false}.$$

As for all $x \in X$, $f(v(x)) \neq f(v(\overline{x}))$, σ is well-defined. We claim that every clause of F is satisfied by σ. Assume, by way of contradiction, some clause $C_i = (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ is not satisfied by σ. Then $f(v(\ell_{i1})) = f(v(\ell_{i2})) = f(v(\ell_{i3})) = 2$, hence $f(c_i) \in \{3, 4\}$ (as c_i is adjacent to all $v(\ell)$), say $f(c_i) = 3$. Therefore $f(c_{i1}) = f(c_{i2}) = f(c_{i3}) = 4$ because the neighbors $v(\ell_{ij})$, c_i and d of c_{ij} have colors $2, 3$ and 1, respectively. But then $f(b_{i1}) = f(b_{i2}) = f(b_{i3}) = 3$, contradicting (1).

Thus, F is satisfied by σ as claimed. □

Lemma 3. *If F is satisfiable then G is 4-colorable.*

Proof. Let σ be a satisfying truth assignment for F. Then a 4-coloring f of G can be defined as follows. First,

- $f(d) := 1$,
- for all $1 \leq i \leq m$, $f(c_i) := 4$,
- for all $\ell \in X \cup \overline{X}$, $f(v(\ell)) := 1$ if $\sigma(\ell) = \texttt{true}$ and $f(v(\ell)) := 2$ otherwise,
- for all $1 \leq i \leq m$, $1 \leq j \leq 3$, $f(c_{ij}) := 2$ if $\sigma(\ell_{ij}) = \texttt{true}$ and $f(c_{ij}) := 3$ otherwise.

Next, as each clause $C_i = (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ has at least one true literal by σ, we can complete our 4-coloring f as follows: For each $1 \leq i \leq m$, let ℓ_{i,j_i} be a true literal in C_i for some $1 \leq j_i \leq 3$. Then set

- $f(b_{ij_i}) := 3$, $f(b_{ij}) := 4$ for $j \in \{1, 2, 3\} \setminus \{j_i\}$,
- $f(a_{ij_i}) := 4$, and the two vertices in $\{a_{i1}, a_{i2}, a_{i3}\} \setminus \{a_{ij_i}\}$ are colored with color 2 and color 3.

It is easy to verify that f is indeed a proper 4-coloring of G. Lemma 3 is proved. □

From Lemmas 1–3 we conclude:

Theorem 4. 4-Col *is NP-complete on P_9-free graphs.* □

Let G be a graph and let H be the graph obtained from G and a new vertex v by adding all the edges between v and vertices of G. Then G is P_t-free if and only if H is P_t-free, and G is k-colorable if and only if H is $(k+1)$-colorable. Hence, by Theorem 4, k-Col is NP-complete on P_t-free graphs for all $k \geq 4$ and all $t \geq 9$.

3. 4-coloring (P_5, C_5)-free graphs

In this section we will show the following result.

Theorem 5. 4-Col *is solvable in polynomial time on (P_5, C_5)-free graphs.*

Proof. Let G be a connected (P_5, C_5)-free graph. We first use the polynomial-time algorithm in [3] to distinguish two cases, according to whether or not G is perfect. If G is perfect, one can even optimally color G in polynomial time (cf. [7]). So, we may assume that G is not perfect.

If G contains a complete graph K_5 on five vertices then G clearly cannot be colored with 4 colors, hence we can assume further that G is K_5-free. By the result in [11] (or by the SPGT [2]), G must contain an induced C_t or $\overline{C_t}$ for some odd integer $t \geq 5$. As G is (P_5, C_5)-free, and $\overline{C_t}$, $t \geq 9$, are not 4-colorable, G therefore must contain an induced $\overline{C_7}$.

Consider an arbitrary induced $\overline{C_7}$, say C, in G. Clearly, G is 4-colorable if and only if there exists a 4-coloring of C that can be extended to a 4-coloring of G. We now explain how to decide in polynomial time if a 4-coloring of C

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can be extended to a 4-coloring of G. As there exist at most 4^7 4-colorings of C, deciding if G is 4-colorable therefore can be done in polynomial time.

Call a vertex $v \in G \setminus C$ a *p-vertex* (with respect to C) if $|N(v) \cap C| = p$. Note that we may assume that there exist no seven-vertices (otherwise G cannot be 4-colorable). Also, there exist no one-vertices (otherwise G would contain an induced P_5) and no two-vertices (otherwise G would contain an induced P_5 or C_5).

Let F denote the set of all five- and six-vertices, let S be the set of all three- and four-vertices, and let R denote the set of all zero-vertices. Then, by the facts above, $V(G) = V(C) \cup F \cup S \cup R$. Furthermore,

$$\text{no vertex in } R \text{ is adjacent to a vertex in } S, \tag{3}$$

otherwise G would contain an induced P_5. We also need the following important fact.

$$\begin{array}{l}\text{For each connected component } A \text{ of } G[R], \text{ every two vertices} \\ \text{in } A \text{ have the same (non-empty) neighborhood in } F.\end{array} \tag{4}$$

To see (4) note first that by the connectedness of G and by (3), $N(A) \subseteq F$ is non-empty. Now, if some vertex $v \in F$ is adjacent to $a_1 \in A$ but non-adjacent to $a_2 \in A$ then, by the connectedness of A, we may assume that a_1 and a_2 are adjacent. But then a_1, a_2, v together with a neighbor and a non-neighbor of v in C induce a P_5. Thus, (4) holds. Note that (4) implies that $N(x) \cap F = N(A)$ for all $x \in A$.

Fix a 4-coloring f of C, and call a vertex $v \in G \setminus C$ *f-bad* if $N(v) \cap C$ contains all 4 colors under f. Note that we may assume further that G has no f-bad vertices, otherwise we conclude that f cannot be extended to a 4-coloring of the whole graph G. Thus, the following facts are easy to check.

$$\text{For all } v \in F, |f(N(v) \cap C)| = 3, \tag{5}$$

and

$$\text{For all } v \in S, 2 \le |f(N(v) \cap C)| \le 3. \tag{6}$$

Let $S_1 \subseteq S$ be the set of all $v \in S$ with $N(v) \cap C$ having exactly 3 colors under f, and set $S_2 := S \setminus S_1$. Thus, by (6), S_2 contains all vertices of S with $N(v) \cap C$ having exactly 2 colors under f.

We now decide, in polynomial time, if f can be extended to a 4-coloring of G as follows. First, by (5) and (6), extend f in $F \cup S_1$ by coloring the vertices $v \in F \cup S_1$ with the remaining 4th color outside $f(N(v) \cap C)$. We denote the 4-coloring after this extension again by f. Clearly, if f is not a proper 4-coloring of $G[C \cup F \cup S_1]$ then f is not extendable to a 4-coloring of G. Thus, we may assume that f is a proper 4-coloring so far, and the colors of vertices in $C \cup F \cup S_1$ have already been fixed.

Next, extend f in S_2 by solving the following 2-SAT instance (which can be done in polynomial time, see, e.g., [5]): For each $v \in S_2$ create two Boolean variables x_i^v, x_j^v where i and j are the two different colors outside $f(N(v) \cap C)$, and introduce the two-literal clause $(x_i^v \vee x_j^v)$ (meaning that $x_i^v = 1$ if v is assigned to color i, $x_i^v = 0$ otherwise). For adjacent vertices $u, v \in S_2$ introduce two two-literal clauses $(\neg x_i^u \vee \neg x_i^v) \wedge (\neg x_j^u \vee \neg x_j^v)$. Finally, for each $u \in F \cup S_1$ adjacent to a vertex $v \in S_2$ with $f(u) = k \notin f(N(v) \cap C)$, create a Boolean variable x_k^u, and introduce the one-literal clause (x_k^u) and the two-literal clause $(\neg x_k^u \vee \neg x_k^v)$. It is easy to see that f is extendable in S_2 if and only if this 2-SAT instance is satisfiable.

Last, assuming f is a proper 4-coloring of $G \setminus R$, we now decide if f can be extended in R as follows. Consider a connected component A of $G[R]$. If $|f(N(A))| = 4$, then, by (4), f clearly cannot be extended in A. So, we may assume that $|f(N(A))| \le 3$. Then f can be extended in A if and only if A is $(4 - |f(N(A))|)$-colorable. Indeed, if A is $(4 - |f(N(A))|)$-colorable then color the vertices in A with the $4 - |f(N(A))|$ colors not appearing in $f(N(A))$ to extend f in A. Conversely, if f is extendable to a proper 4-coloring in A then, by (4), $f(N(A)) \cap f(A) = \emptyset$, hence $|f(A)| \le 4 - |f(N(A))|$ and thus A is $(4 - |f(N(A))|)$-colorable. Since $4 - |f(N(A))| \le 3$, deciding if A is $(4 - |f(N(A))|)$-colorable can be done in polynomial time ([15,12]).

Since $N(A) \subseteq F$, the last extension of f in R is independent from the 2-SAT solution used in extending f in S_2. Therefore, given a 4-coloring f of C, we can decide in polynomial time if f can be extended to a 4-coloring of G, and the proof of Theorem 5 is complete. □

4. Conclusion

In this note we have shown that, for any fixed integers $t \geq 9$ and $k \geq 4$, deciding if a graph without induced paths on t vertices is k-colorable is an NP-complete problem, and that deciding if a graph without induced paths or cycles on five vertices is 4-colorable is a polynomial-time solvable problem.

Note added in proof. While revising the submitted version of this paper, Chính T. Hoàng informed us that he and his co-authors have shown in [9] that, for any fixed k, k-COL can be solved in polynomial time on P_5-free graphs. This very recent result and Theorem 4 give the current status of the computational complexity of k-COL on P_t-free graphs in Table 1.

Table 1
LIN, P, NP-c, '?' means that the complexity status of the corresponding k-COL problem on P_t-free graphs is linear, polynomial, NP-complete, unknown, respectively

$k \setminus t$	4	5	6	7	8	9	10	11	…
3	LIN	$O(n^\alpha)$	$O(mn^\alpha)$	?	?	?	?	?	…
4	LIN	P	?	?	?	NP-c	NP-c	NP-c	…
5	LIN	P	?	?	NP-c	NP-c	NP-c	NP-c	…
6	LIN	P	?	?	NP-c	NP-c	NP-c	NP-c	…
7	LIN	P	?	?	NP-c	NP-c	NP-c	NP-c	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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