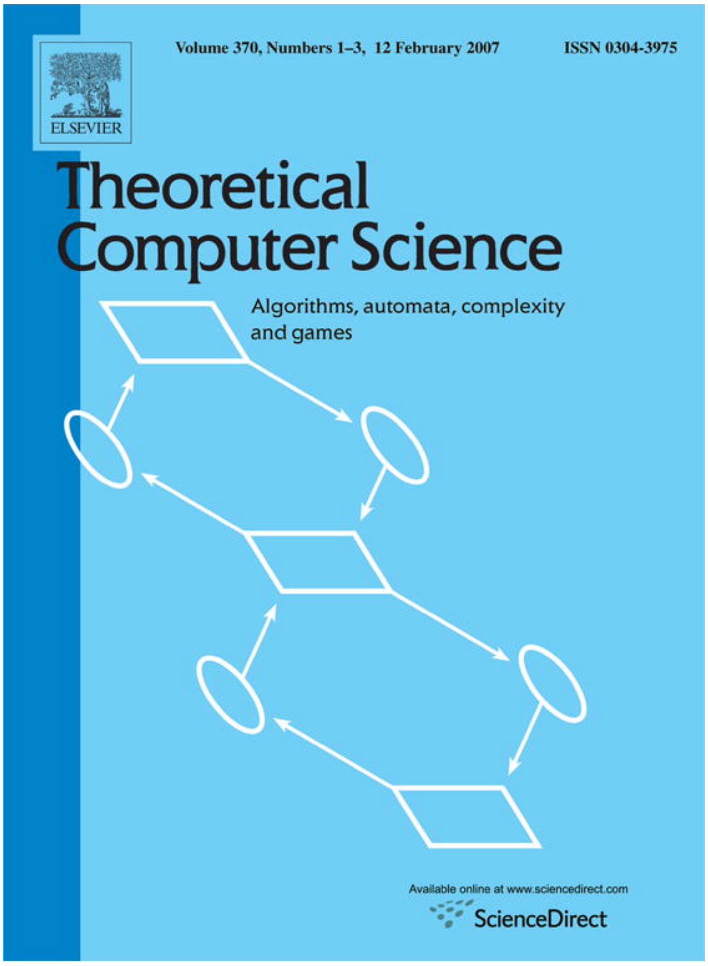
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New applications of clique separator decomposition for the Maximum Weight Stable Set problem

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Abstract

Graph decompositions such as decomposition by clique separators and modular decomposition are of crucial importance for designing efficient graph algorithms. Clique separators in graphs were used by Tarjan as a divide-and-conquer approach for solving various problems such as the Maximum Weight Stable Set (MWS) problem, Colouring and Minimum Fill-in. The basic tool is a decomposition tree of the graph whose leaves have no clique separator (so-called *atoms*), and the problem can be solved efficiently on the graph if it is efficiently solvable on its atoms. We give new examples where the clique separator decomposition works well for the MWS problem; our results improve and extend various recently published results. In particular, we describe the atom structure for some new classes of graphs whose atoms are P_5-free (the P_5 is the induced path with five vertices) and obtain new polynomial time results for the MWS problem. The complexity of this problem on the class of P_5-free graphs is still unknown.

Keywords: Maximum Stable Set problem; Clique separators in graphs; Graph decompositions; P_5-free graphs; Efficient graph algorithms

1. Introduction

In an undirected graph $G = (V, E)$, a *stable* (or *independent*) vertex set is a subset of mutually nonadjacent vertices. The *Maximum Weight Stable* (or *Independent*) *Set* (*MWS*) problem asks for a stable set of maximum weight sum for a vertex weight function w on V. The *MS problem* is the MWS problem where all vertices have the same weight. Let $\alpha_w(G)$ $(\alpha(G))$ denote the maximum weight (maximum cardinality) of a stable vertex set in G.

The M(W)S problem is one of the fundamental algorithmic graph problems which frequently occurs as a subproblem in models in computer science and operations research. It is closely related to the Vertex Cover problem and to the Maximum Clique problem in graphs (for an extensive survey on the last one, see [9], which, at the same time, can be seen as a survey on the MWS and the Vertex Cover problem; however, since 1999, there are many new results on this topic).

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The MWS problem is known to be NP-complete in general and remains NP-complete even on very restricted instances such as $K_{1,4}$-free graphs [42], ($K_{1,4}$,diamond)-free graphs [24], very sparse planar graphs of maximum degree three and graphs not containing cycles below a certain length [47], in particular on triangle-free graphs [48].

On the other hand, it is known to be solvable in polynomial time on many graph classes by various techniques such as polyhedral optimization, augmenting, struction and other transformations, modular decomposition, bounded clique-width and bounded treewidth, reduction of α-redundant vertices, to mention some basic techniques; for a small selection of papers dealing with particular graph classes and such techniques for M(W)S, see [2–4,7,8,10–22,25–27,30–40,42,44–46,50].

Many of these papers deal with subclasses of P_5-free graphs, motivated by the fact that the complexity of the M(W)S problem for P_5-free graphs (and even for (P_5, C_5)-free graphs) is still unknown (for all other 5-vertex graphs H, MS is solvable in polynomial time on (P_5, H)-free graphs).

For $2K_2$-free graphs, however, the following is known:

Farber [28] has shown that a $2K_2$-free graph $G = (V, E)$ contains at most n^2 inclusion-maximal independent sets, $n = |V|$. Thus, the MWS problem on these graphs can be solved in time $\mathcal{O}(n^3 m)$, $m = |E|$, since [23,52] gave a procedure that generates all maximal independent sets in a graph in $\mathcal{O}(nm)$ time per generated set.

This result has been generalized to $\ell \geq 2$: ℓK_2-free graphs have at most $n^{2\ell-2}$ inclusion-maximal independent sets [1,5,29,49], and thus, MWS is solvable on ℓK_2-free graphs in time $\mathcal{O}(n^{2\ell-1} m)$.

Let Π denote a graph property. A graph is *nearly* Π if for each of its vertices v, the subgraph induced by the antineighbourhood $\overline{N}(v)$, i.e. by the set of the nonneighbours of v has property Π. (Note that this notion appears in the literature in many variants, e.g. as nearly bipartite graphs [6].)

Obviously, the MWS problem on a graph G with vertex weight function w can be reduced to the same problem on antineighborhoods $\overline{N}(v)$ of vertices v in the following way:

$$\alpha_w(G) = \max\{w(v) + \alpha_w(G[\overline{N}(v)]) \mid v \in V\}.$$

Thus, whenever MWS is solvable in time T on a class with property Π then it is solvable on nearly Π graphs in time $n \cdot T$. For example, Corneil, Perl and Stewart [25] gave a linear time algorithm for MWS on cographs along the cotree of such a graph. Thus, MWS is solvable in time $\mathcal{O}(nm)$ on nearly cographs. This simple fact, for example, immediately implies Theorem 1 of [30] (which is formulated in [30] for the Maximum Clique problem and shown there in a more complicated way). For other examples where this approach is helpful, see [13].

A famous divide-and-conquer approach by using clique separators (also called clique cutsets) is described by Tarjan in [51] (see also [53]). For various problems on graphs such as Minimum Fill-in, Coloring, Maximum Clique, and the MWS problem, it works well in a bottom-up way along a clique separator tree (which is not uniquely determined but can be constructed in polynomial time for a given graph). The leaves of such a tree, namely the subgraphs not containing clique separators are called *atoms* in [51]. Whenever MWS is solvable in time T on the atoms of a graph G, it is solvable in time $n^2 \cdot T$ on G. However, few examples are known where this approach could be applied for obtaining a polynomial time MWS algorithm on a graph class.

Modular decomposition of graphs is another powerful tool. The decomposition tree is uniquely determined and can be found in linear time [41]. The prime nodes in the tree are the subgraphs having no homogeneous sets (definitions are given later). Again, various problems can be solved in time $\mathcal{O}(T)$ bottom-up along the modular decomposition tree, among them Maximum Clique, and the MWS problem, provided they can be solved in time T on the prime nodes. In [13], it was shown that a combination of both decompositions is helpful for the MWS problem: If MWS is solvable in time T on prime atoms (i.e. prime subgraphs without clique cutset) of the graph G then it is solvable in time $n^2 \cdot T$ on G.

One of the examples where the clique separator approach works well is given by Alekseev in [3] showing that atoms of (P_5, Q)-free graphs are $3K_2$-free which implies that the MWS problem is solvable in time $\mathcal{O}(n^7 m)$ on this graph class (see Fig. 1 for the graph Q). In [13], it was shown that atoms of (P_5, Q)-free graphs are either nearly $(P_5, \overline{P_5}, C_5)$-free or specific (i.e. a simple type of graphs for which the MWS problem can be solved in the obvious way). This leads to an $\mathcal{O}(n^4 m)$ time algorithm for MWS on (P_5, Q)-free graphs which improves and extends Alekseev's result on these graphs [3].

Our main results in this paper are the following ones:

(i) Prime atoms of (P_5, F_1)-free graphs are $3K_2$-free (see Fig. 1 for the F_1). By [13], this implies polynomial time for MWS on (P_5, F_1)-free graphs which extends corresponding polynomial time results on (P_5, Q)-free graphs,

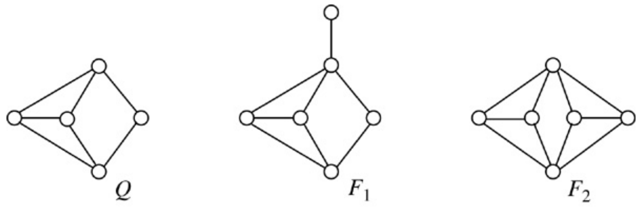

Fig. 1. The graphs Q, F_1 and F_2.

on (P_5,co-chair)-free graphs [22], and on (P_5, P)-free graphs [13,20,39] (note, however, that the time bound for (P_5, F_1)-free graphs is much worse than on the last two subclasses mentioned here).

(ii) Atoms of (P_5, F_2)-free graphs are $4K_2$-free (see again Fig. 1 for the F_2). This also extends the result on (P_5, Q)-free graphs.

(iii) Finally, we show that for every fixed k, MS can be solved in polynomial time for (P_5, H_k)-free graphs (see Fig. 3 for the H_k which extends F_1 and F_2). Note that for each $k \geq 2$, the class of (P_5, H_k)-free graphs contains all (P_5, F_1)-free and all (P_5, F_2)-free graphs.

These results give new examples for the power of clique separators.

2. Basic notions

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. Let $V(G) = V$ denote the vertex set of graph G. For a vertex $v \in V$, let $N(v) = \{u \mid uv \in E\}$ denote the (*open*) *neighbourhood* of v in G, let $N[v] = \{v\} \cup \{u \mid uv \in E\}$ denote the (*closed*) *neighbourhood* of v in G, and for a subset $U \subseteq V$ and a vertex $v \notin U$, let $N_U(v) = \{u \mid u \in U, uv \in E\}$ denote the *neighbourhood* of v with respect to U. The *antineighbourhood* $\overline{N}(v)$ is the set $V \setminus N[v]$ of vertices different from v which are nonadjacent to v. We also write $x \sim y$ for $xy \in E$ and $x \not\sim y$ for $xy \notin E$.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. $G[U]$ is *co-connected* if $\overline{G}[U]$ is connected.

Let $\mathcal{F}$ denote a set of graphs. A graph G is $\mathcal{F}$*-free* if none of its induced subgraphs is in $\mathcal{F}$.

A vertex set $U \subseteq V$ is *stable* (or *independent*) in G if the vertices in U are pairwise nonadjacent. For a given graph with vertex weights, the Maximum Weight Stable Set (MWS) problem asks for a stable set of maximum vertex weight.

Let $\overline{G} = (V, \overline{E})$ denote the *complement graph* of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$. Let K_ℓ denote the clique with ℓ vertices, and let ℓK_1 denote the stable set with ℓ vertices. K_3 is called *triangle*.

Disjoint vertex sets X, Y *form a join*, denoted by $X \textcircled{1} Y$ (*co-join*, denoted by $X \textcircled{0} Y$) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds. We will also say that X has a join to Y, that there is a join between X and Y, or that X and Y are connected by join (and similarly for co-join). Subsequently, we will consider join and co-join also as operations, i.e. the co-join operation for disjoint vertex sets X and Y is the disjoint union of the subgraphs induced by X and Y (without edges between them), and the join operation for X and Y consists of the co-join operation for X and Y followed by adding all edges $xy \in E$, $x \in X$, $y \in Y$.

A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$ or $zx \notin E$ and $zy \in E$. We also say that *a vertex z distinguishes a vertex set $U \subseteq V$, $z \notin U$*, if z has a neighbour and a non-neighbor in U. The following facts are well-known and easy to see.

Observation 1. *Let $v \in G[V \setminus U]$ distinguish U.*

(i) *If $G[U]$ is connected, then there exist two adjacent vertices $x, y \in U$ such that $v \sim x$ and $v \not\sim y$.*

(ii) *If $G[U]$ is co-connected, then there exist two nonadjacent vertices $x, y \in U$ such that $v \sim x$ and $v \not\sim y$.*

A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e. every vertex $v \in V \setminus M$ has either a join or a co-join to M. A module is *trivial* if it is $\emptyset$, $V(G)$ or a one-element vertex set. A nontrivial module is also called a *homogeneous set*. A graph G is *prime* if it contains only trivial modules.

The notion of module plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms — see, e.g. [43] for modular decomposition of discrete structures and its algorithmic use and [41] for a linear-time algorithm constructing the modular decomposition tree of a given graph.

A *clique separator* or *clique cutset* in a connected graph G is a clique C such that $G[V \setminus C]$ is disconnected. An *atom* of G is a subgraph of G without clique cutset. See [51] for some algorithmic aspects of the clique separator decomposition.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k-1$ edges. For $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges.

The $2K_2$ is the $\overline{C_4}$. More generally, the ℓK_2 consists of 2ℓ vertices, say, $x_1, \ldots, x_\ell$, $y_1, \ldots, y_\ell$ and edges $x_1 y_1, \ldots, x_\ell y_\ell$.

3. Minimal cutsets for ℓK_2 in P_5-free graphs

In this section we will collect some useful facts about P_5-free graphs that contain an induced ℓK_2. These facts will be used to prove our main results in Sections 4 and 5; they represent a more detailed investigation of the background of Alekseev's theorem on (P_5, Q)-free graphs in [3], mentioned in the introduction.

Let $\ell \geq 2$ be an integer, and let G be a P_5-free graph containing an induced $H = \ell K_2$ with $E(H) = \{e_1, e_2, \ldots, e_\ell\}$. Let $S \subseteq V(G) \setminus V(H)$ be an inclusion-minimal vertex set such that, for $i \neq j$, e_i and e_j belong to distinct connected components of $G[V \setminus S]$. S is also called a *minimal cutset* for H. For $1 \leq i \leq \ell$, let H_i be the connected component of $G[V \setminus S]$ containing the edge e_i.

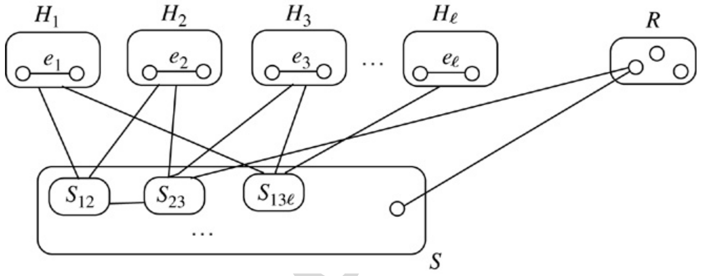

Fig. 2. The minimal cutset S for ℓK_2 and its partition into subsets S_L illustrated.

Observation 2. (i) $\forall v \in S$: *$N(v) \cap H_i \neq \emptyset$ and $N(v) \cap H_j \neq \emptyset$ for at least two distinct indices $i, j \in \{1, 2, \ldots, \ell\}$.*
(ii) $\forall v \in S$: *v distinguishes at most one H_i, $i \in \{1, 2, \ldots, \ell\}$.*

Proof. (i): First, note that by the minimality of S, every vertex in S must have a neighbour in $H_1 \cup H_2 \cup \cdots \cup H_\ell$. Now, assume to the contrary that some vertex $v \in S$ has the property that $N(v) \cap H_i \neq \emptyset$ for exactly one $i \in \{1, 2, \ldots, \ell\}$. Then $H_i \cup \{v\}$ and H_j, $j \neq i$, are the connected components of $G[V \setminus (S - v)]$ containing the edge e_i, respectively, e_j $(j \neq i)$, i.e. $S - v$ is a smaller cutset for H which contradicts to the minimality of S.
(ii): Assume to the contrary that some vertex $v \in S$ distinguishes H_i and H_j for indices $i \neq j$. By Observation 1(i), there are adjacent vertices $x, y \in H_i$, $x', y' \in H_j$ such that $v \sim x$, $v \not\sim y$ and $v \sim x'$, $v \not\sim y'$. Then $yxvx'y'$ induce a P_5 in G, a contradiction. □

By Observation 2(i), S can be partitioned into pairwise disjoint subsets S_L as follows. For $L \subseteq \{1, 2, \ldots, \ell\}$, $|L| \geq 2$, let S_L be the subset of S consisting of all vertices in S that have a neighbour in each H_i, $i \in L$, but no neighbour in H_j, $j \in \{1, 2, \ldots, \ell\} \setminus L$. Formally,

$$S_L := \{v \in S \mid (\forall i \in L, N(v) \cap H_i \neq \emptyset) \wedge (\forall j \notin L, N(v) \cap H_j = \emptyset)\}.$$

Set

$$R := V \setminus (S \cup H_1 \cup H_2 \cup \cdots \cup H_\ell).$$

Note that R is the union of all connected components of $G[V \setminus S]$ different from $H_1, H_2, \ldots, H_\ell$. In particular, $R \textcircled{0} (H_1 \cup H_2 \cup \cdots \cup H_\ell)$, and R is possibly nonempty; cf. Fig. 2.

In what follows, L, M, N stand for subsets of $\{1, 2, \ldots, \ell\}$ with at least two elements. Two such subsets are called *incomparable* if each of them is not properly contained in the other. Incomparable sets L, M are *overlapping* if $L \cap M \neq \emptyset$. Note that disjoint sets are mutually incomparable.

Observation 3. *Let L and M be incomparable. Then, for all adjacent vertices $x \in S_L$, $y \in S_M$, $x \textcircled{1} (\bigcup_{i \in L \setminus M} H_i)$ and $y \textcircled{1} (\bigcup_{j \in M \setminus L} H_j)$.*

Proof. Assume to the contrary that x is nonadjacent to a vertex in H_i for some $i \in L \setminus M$. By Observation 1(i), there are adjacent vertices $u, v \in H_i$ such that $x \sim u$ and $x \not\sim v$. Then v, u, x, y, and any neighbour of y in H_j, $j \in M \setminus L$, induce a P_5, a contradiction.

Thus $x \textcircled{1} (\bigcup_{i \in L \setminus M} H_i)$ and, by symmetry, $y \textcircled{1} (\bigcup_{j \in M \setminus L} H_j)$. □

Observation 4. *Let L and M be overlapping. Then*

(i) *$S_L \textcircled{1} S_M$, and*
(ii) *if $S_L \neq \emptyset$ and $S_M \neq \emptyset$ then $S_L \textcircled{1} (\bigcup_{j \in L \setminus M} H_j)$ and $S_M \textcircled{1} (\bigcup_{i \in M \setminus L} H_i)$.*

Proof. Let $i \in L \setminus M$, $j \in L \cap M$, and $k \in M \setminus L$.
(i): Assume to the contrary that a vertex $x \in S_L$ is nonadjacent to a vertex $y \in S_M$. Let $u \in H_j$ be a neighbour of x and $v \in H_j$ be a neighbour of y, and consider a shortest path P in H_j between u and v. Then x, y, P, a neighbour of x in H_i, and a neighbour of y in H_k induce a path with at least five vertices, a contradiction.
(ii): This statement follows from Observation 4(i) and Observation 3. □

Observation 5. *Let M be a proper subset of L. Then for all nonadjacent vertices $x \in S_M$, $y \in S_L$,*

(i) *$y \textcircled{1} (\bigcup_{i \in L \setminus M} H_i)$, and*
(ii) *for all $j \in M$, $N(x) \cap H_j \subseteq N(y) \cap H_j$.*

Proof. (i): If y is nonadjacent to a vertex in H_i for some $i \in L \setminus M$, then, by Observation 1(i), there are adjacent vertices $u, v \in H_i$ such that $y \sim u$, $y \not\sim v$. For $j \in M$, let $u' \in H_j$ be a neighbour of x and $v' \in H_j$ be a neighbour of y. Consider a shortest path P in H_j between u' and v'. Then x, P, y, u, and v induce a path with at least five vertices, a contradiction.
(ii): Assume that, for some $j \in M$, there exists a vertex $u \in N(x) \cap H_j$ which is nonadjacent to y. Let k be an index in $M \setminus \{j\}$. Again, let $u' \in H_k$ be a neighbour of x and let $v' \in H_k$ be a neighbour of y. Consider a shortest path P in H_k between u' and v'. Then u, x, P, y and any vertex in H_i, $i \in L \setminus M$, induce a path with at least five vertices, a contradiction. □

Observation 6. *Let $L \cap N = \emptyset$. If some vertex in S_L is nonadjacent to some vertex in S_N, then for all subsets M overlapping with L and with N, $S_M = \emptyset$.*

Proof. Assume to the contrary that $S_M \neq \emptyset$. By Observation 4, $S_L \textcircled{1} S_M$ and $S_M \textcircled{1} S_N$. Now, if $x \in S_L$ is nonadjacent to $y \in S_N$ then x, y together with any vertex in S_M, any vertex in H_i, $i \in L \setminus M$, and any vertex in H_j, $j \in N \setminus M$, induce a P_5, a contradiction. □

For each subset $L \subseteq \{1, 2, \ldots, \ell\}$ with at least two elements we partition S_L into pairwise disjoint subsets as follows. Let:

$$X_L := \{v \in S_L \mid \forall i \in L, v \textcircled{1} H_i\},$$

and for each $i \in L$,

$$Y_L^i := \{v \in S_L \mid v \text{ distinguishes } H_i\}.$$

By Observation 2(ii),

$$\forall i \in L, Y_L^i \textcircled{1} \Big(\bigcup_{j \in L \setminus \{i\}} H_j \Big) \quad \text{and} \quad S_L = X_L \cup \bigcup_{i \in L} Y_L^i.$$

Observation 7. *If $|L| \geq 3$ then for all distinct $i, j \in L$, $Y_L^i \textcircled{1} Y_L^j$.*

Proof. Assume that for some $i, j \in L$ with $i \neq j$, a vertex $x \in Y_L^i$ is nonadjacent to a vertex $y \in Y_L^j$. Then there is a vertex $v \in H_i$ such that $x \not\sim v$, and there is a vertex $v' \in H_j$ such that $y \not\sim v'$. Recall that $x \sim v'$ and $y \sim v$. Then v, v', x, y, and any vertex in H_k, $k \in L \setminus \{i, j\}$, induce a P_5 in G, a contradiction. □

Observation 8. *If* $|L| \geq 3$ *and* G *is* F_1*-free or* F_2*-free then* $X_L \textcircled{1} (S_L \setminus X_L)$.

Proof. Assume that a vertex $x \in X_L$ is nonadjacent to a vertex $y \in S_L \setminus X_L$, say $y \in Y_L^i$ for some $i \in L$. Let $j \neq k$ be two indices in $L \setminus \{i\}$. Then x, y together with an edge in H_j, an edge in H_k, and a vertex in $H_i \setminus N(y)$ induce a subgraph in G containing both F_1 and F_2, a contradiction. □

4. Prime atoms of (P_5, F_1)-free graphs are $3K_2$-free

In this section we show:

Theorem 1. *Prime* (P_5, F_1)*-free graphs without clique cutset are* $3K_2$*-free.*

Proof. Assume that the prime (P_5, F_1)-free graph G contains an induced subgraph $H = 3K_2$. We are going to show that G contains a clique cutset. We use the notations and definitions in Section 3 for $\ell = 3$. We also write S_{ij} for $S_{\{i,j\}}$, $i, j \in \{1, 2, 3\}$, and S_{123} for $S_{\{1,2,3\}}$.

Claim 4.1. *For all* $i, j \in \{1, 2, 3\}$ *with* $i \neq j$, $S_{123} \textcircled{1} S_{ij}$ *holds.*

Proof of Claim 4.1. Assume that a vertex $x \in S_{ij}$ is nonadjacent to a vertex $y \in S_{123}$. By Observation 2(ii), x distinguishes at most one of H_i, H_j, and thus has a join to the other, say $x \textcircled{1} H_i$. Then by Observation 5, the vertices x, y, an edge in H_i, a vertex in $N(x) \cap H_j$, and a vertex in H_k, $k \in \{1, 2, 3\} \setminus \{i, j\}$, induce an F_1, a contradiction which shows Claim 4.1.

Claim 4.2. S_{123} *is a clique.*

Proof of Claim 4.2. We first show that X_{123} is a clique. Assume to the contrary that there exists a nontrivial connected component C in $\overline{G}[X_{123}]$. As G is prime, there exists some vertex $a \notin C$ distinguishing C. Clearly, $a \notin X_{123}$. By Observation 8, Claim 4.1, and by definition of X_{123}, $a \in R$. Moreover, by Observation 1(ii), there are nonadjacent vertices c_1, c_2 in C such that $a \sim c_1$, $a \not\sim c_2$. Then c_1, c_2, a together with an edge in H_1, and any vertex in H_2 induce an F_1, a contradiction. Thus, X_{123} must be a clique. By similar arguments, it can be seen that, for each $i = 1, 2, 3$, Y_{123}^i is a clique. Then, by Observations 7 and 8, S_{123} is a clique and Claim 4.2 follows.

Now, if two of S_{ij}, S_{ik}, S_{jk} are cliques, say S_{ij} and S_{ik}, then by Observation 4(i), Claims 4.1 and 4.2, $S_{ij} \cup S_{ik} \cup S_{123}$ is a clique. This clique clearly separates H_i and H_j, and we are done.

Thus, we may assume that (at least) two of S_{ij}, S_{ik}, S_{jk} are not cliques, say S_{ij} and S_{ik}. Note that by Observation 4(ii), $S_{ij} \textcircled{1} H_j$ and $S_{ik} \textcircled{1} H_k$.

As S_{ij} is not a clique, there exists a nontrivial connected component C in $\overline{G}[S_{ij}]$. As G is prime, there exists a vertex $a \notin C$ distinguishing C. By Observation 4 and Claim 4.1, $a \in H_i \cup R$. By Observation 1(ii), there are two nonadjacent vertices $c_1 \neq c_2$ in C such that $a \sim c_1, a \not\sim c_2$. Moreover, $a \textcircled{1} S_{ik}$: For, if a is nonadjacent to a vertex $x \in S_{ik}$ then c_1, c_2, a, x together with an edge in H_j would induce an F_1. Likewise, there exists a nontrivial connected component C' in $\overline{G}[S_{ik}]$, a vertex $b \in H_i \cup R$, and two nonadjacent vertices $c_1' \neq c_2'$ in C' such that $b \sim c_1', b \not\sim c_2'$, and $b \textcircled{1} S_{ij}$.

Now, if $a \not\sim b$ then a, b, c_1, c_2 together with an edge in H_j induce an F_1, a contradiction. If $a \sim b$ then a, b, c_1, c_2, c_2' together with a vertex in H_k induce an F_1, which is again a contradiction. This shows Theorem 1. □

By Theorem 1, prime (P_5, F_1)-free atoms are $3K_2$-free, hence MWS can be solved in time $\mathcal{O}(n^5 m)$ on prime (P_5, F_1)-free atoms with n vertices and m edges. Combining with the time bound for MWS via clique separators, we obtain:

Corollary 1. *The MWS problem can be solved in time* $\mathcal{O}(n^7 m)$ *for* (P_5, F_1)*-free graphs.*

5. Atoms of (P_5, F_2)-free graphs are $4K_2$-free

In this section we show:

Theorem 2. *(P_5, F_2)-free graphs without clique cutset are $4K_2$-free.*

Proof. Assume that the (P_5, F_2)-free graph G contains an induced subgraph $H = 4K_2$. We are going to show that G contains a clique cutset. We use the notations and definitions in Section 3 for $\ell = 4$. Thus, L, M, N stand for subsets of $\{1, 2, 3, 4\}$ with at least two elements.

Claim 5.1.

(i) *$\forall L$, X_L is a clique. Moreover, if $|L| \geq 3$ then S_L is a clique.*
(ii) *Let $|L| \geq 3$. Then, for all $M \neq L$, X_M①S_L. Moreover, if $|M| \geq 3$ then S_M①S_L.*

Proof of Claim 5.1. (i): If X_L is not a clique then two nonadjacent vertices in X_L together with an edge in H_i and an edge in H_j, $i, j \in L$, $i \neq j$, induce an F_2. Let $|L| \geq 3$. Then, as before, Y_L^k are cliques, $k \in L$. By Observations 7 and 8, S_L is a clique.
(ii): Recall that all considered subsets of $\{1, 2, 3, 4\}$ have at least two elements. Hence $M \cap L \neq \emptyset$ whenever $|L| \geq 3$. Now, if M and L are overlapping then S_M①S_L by Observation 4(i). If M and L are not overlapping then in the case $L \subset M$, S_M①S_L holds by Observation 2(ii). The same argument holds in the case $M \subset L$ and $|M| = 3$. Now assume that $M \subset L$ and $|M| = 2$. Hence, if some $x \in X_M$ is nonadjacent to some $y \in S_L$ then by Observation 5(ii), x, y together with an edge in H_i and an edge in H_j (where $M = \{i, j\}$) induce an F_2 which shows Claim 5.1.

Let $Q := \bigcup_{|L| \geq 3} S_L$. By Claim 5.1, Q is a clique. If there exists some $i \in \{1, 2, 3, 4\}$ such that, for all 2-element subsets M containing i, $S_M = \emptyset$, then Q is a clique cutset in G, separating H_i and H_j, $i \neq j$.

So, we may assume

$$\forall i \in \{1, 2, 3, 4\}, \text{ there exists a 2-element subset } M \subset \{1, 2, 3, 4\} \text{ with } i \in M \text{ and } S_M \neq \emptyset. \tag{1}$$

In particular, there exist at least two distinct 2-element subsets M, N with $S_M \neq \emptyset \neq S_N$. We distinguish between two cases.

Case 1. There exist disjoint 2-element subsets M, N with $S_M \neq \emptyset$ and $S_N \neq \emptyset$.

We first assume that there exist such subsets M and N such that some vertex in S_M is nonadjacent to some vertex in S_N. Then, by Observation 6, for all 2-element subsets L different from M and N, $S_L = \emptyset$. Moreover, by Observation 3, there exists no edge connecting S_N and $S_M \setminus X_M$. Now, by Claim 5.1, $X_M \cup Q$ is a clique. This clique separates H_i, $i \in N$, and H_j, $j \in M$, because any shortest path in $G \setminus (X_M \cup Q)$ connecting H_i and H_j must use at least one vertex in R, hence must contain an induced P_5. Thus, for all disjoint 2-element subsets I, J with $S_I \neq \emptyset \neq S_J$, S_I①S_J. Then,

for *all* 2-element subsets $I \neq J$ with $S_I \neq \emptyset \neq S_J$, S_I①S_J:

The assertion is clear for $I \cap J = \emptyset$; if $I \cap J \neq \emptyset$, it follows from Observation 4(i).

Now consider disjoint 2-element subsets M, N with $S_M \neq \emptyset \neq S_N$. As S_M①S_N, $S_M = X_M$ and $S_N = X_N$ by Observation 3. Moreover,

for all other 2-element subsets L, $S_L = X_L$, too,

because S_M①S_L and S_N①S_L, hence by Observation 3 again, S_L① $\left(\bigcup_{i \in L \setminus M} H_i \cup \bigcup_{j \in L \setminus N} H_j\right)$. This and the fact $(L \setminus M) \cup (L \setminus N) = L$ show $S_L = X_L$.

Therefore, by Claim 5.1, $S = \bigcup_{|L| \geq 2} S_L$ is a clique. This clique clearly separates H_i and H_j, $i \neq j$. Case 1 is settled.

Case 2. For all disjoint 2-element subsets M, N, $S_M = \emptyset$ or $S_N = \emptyset$.

By (1), we may assume without loss of generality that

S_{12}, S_{13} and S_{14} are nonempty,

say. Then, by the hypothesis of this case,

for all other 2-element subsets L, $S_L = \emptyset$,

and, recall that by Observation 4,

$$\forall i, 2 \leq i \leq 4, \forall L \text{ with } 1 \notin L \text{ or } i \notin L\text{: } S_{1i}\,\textcircled{1}\,S_L, \tag{2}$$

$$\forall i, 2 \leq i \leq 4 : S_{1i}\,\textcircled{1}\,H_i, \tag{3}$$

and

$$\forall i \neq j, 2 \leq i, j \leq 4 : S_{1ij}\,\textcircled{1}\,(H_i \cup H_j). \tag{4}$$

Claim 5.2. (i) $S_{12} \cup S_{13} \cup S_{14}$ *is a clique.*
(ii) $(S_{12} \cup S_{13} \cup S_{14})\,\textcircled{1}\,S_{1234}$.

Proof of Claim 5.2. (i): By (2), it remains to show that, for each $i = 2, 3, 4$, S_{1i} is a clique. Indeed, by (1) and (2), two nonadjacent vertices in S_{1i} together with an edge in H_i, a vertex in S_{1j}, and a vertex in S_{1k} for $\{i, j, k\} = \{2, 3, 4\}$, induce an F_2.
(ii): Fix an $i \in \{2, 3, 4\}$, and assume that some vertex $x \in S_{1i}$ is nonadjacent to some vertex $y \in S_{1234}$. Then, by (3) and Observation 5(ii), $y\,\textcircled{1}\,(H_i \cup H_j \cup H_k)$ where $\{i, j, k\} = \{2, 3, 4\}$. Moreover, $y\,\textcircled{1}\,(S_{1j} \cup S_{1k})$: For, if y is nonadjacent to a vertex $a \in S_{1j}$, say, then a, x, y together with any vertex in H_i, and any vertex in H_k induce a P_5. Now, x, y together with an edge in H_i, any vertex in S_{1j}, and any vertex in S_{1k} induce an F_2 which shows Claim 5.2.

By (2) and Claims 5.1 and 5.2, if for some $i \in \{2, 3, 4\}$, $S_{1i}\,\textcircled{1}\,(S_{1ij} \cup S_{1ik})$ (where $\{i, j, k\} = \{2, 3, 4\}$), then $\bigcup_{i \in L} S_L$ is a clique. This clique clearly separates H_1 and H_i, and we are done.

So, let us assume that

$S_{12} \cup (S_{123} \cup S_{124})$, $S_{13} \cup (S_{123} \cup S_{134})$, and $S_{14} \cup (S_{124} \cup S_{134})$ all are not cliques.

We will obtain a contradiction. Letting $\{i, j, k\} = \{2, 3, 4\}$, we first show:

Let $x \in S_{1i}$, $y \in S_{1j}$, $z \in S_{1k}$, and suppose that z is nonadjacent to some vertex in $S_{1ik} \cup S_{1jk}$.
Then $N_{H_1}(x) \cap N_{H_1}(y) = \emptyset$. (5)

Proof of (5). Note first that, by (2), x, y, z are pairwise adjacent, and by (3), $x\,\textcircled{1}\,H_i$, $y\,\textcircled{1}\,H_j$, $z\,\textcircled{1}\,H_k$.

Let $u \in S_{1ik} \cup S_{1jk}$ be a vertex nonadjacent to z. Then, by (4), $u\,\textcircled{1}\,H_k$, hence $x \not\sim u$ or $y \not\sim u$, otherwise x, y, z, u, and an edge in H_k would induce an F_2. On the other hand, by (2), $x \sim u$ (if $u \in S_{1jk}$) or $y \sim u$ (if $u \in S_{1ik}$).

Now, assume to the contrary that there exists some vertex $v \in N_{H_1}(x) \cap N_{H_1}(y)$. Then, by Observation 5(ii), $u \sim v$. Hence x, y, u, v, and an edge in H_i (if $u \in S_{1ik}$) or an edge in H_j (if $u \in S_{1jk}$) induce, by (4), an F_2. This contradiction proves (5).

Now, fix a vertex $a \in S_{12}$ nonadjacent to some vertex in $S_{123} \cup S_{124}$, a vertex $b \in S_{13}$ nonadjacent to some vertex in $S_{123} \cup S_{134}$, and a vertex $c \in S_{14}$ nonadjacent to some vertex in $S_{124} \cup S_{134}$. Then, by (5), $N_{H_1}(a) \cap N_{H_1}(b) = N_{H_1}(a) \cap N_{H_1}(c) = N_{H_1}(b) \cap N_{H_1}(c) = \emptyset$.

Consider a shortest path P in H_1 connecting $N_{H_1}(a)$ and $N_{H_1}(b)$. If P is an edge, P, a, c, and any vertex in H_4 induce a P_5. If P is not an edge, P, a, and any vertex in H_2 induce a path of length at least five. This final contradiction settles Case 2, and Theorem 2 follows. □

By Theorem 2, (P_5, F_2)-free atoms are $4K_2$-free, hence MWS can be solved in time $\mathcal{O}(n^7 m)$ on (P_5, F_2)-free atoms. Combining again with the clique separator time bound for MWS, we obtain:

Corollary 2. *Maximum Weight Stable Set can be solved in time* $\mathcal{O}(n^9 m)$ *for* (P_5, F_2)*-free graphs.*

6. Maximum Stable Set problem in (P_5, H_k)-free graphs

One way in trying to show that the Maximum Weight Stable Set problem can be solved in polynomial time on a large class of P_5-free graphs containing both classes of (P_5, F_1)-free graphs and of (P_5, F_2)-free graphs is to consider the class of (P_5, H_k)-free graphs, for each fixed integer $k \geq 2$; see Fig. 3.

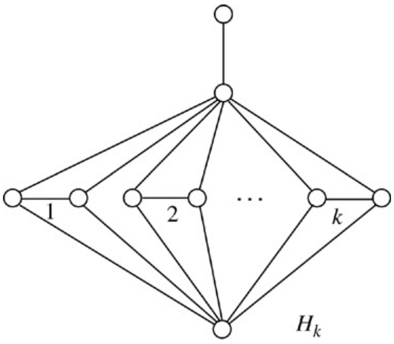

Fig. 3. The graph H_k.

Unfortunately, the technique used in this paper cannot be directly applied for (P_5, H_k)-free graphs. Namely, for each fixed $\ell \geq 3$, there exist *prime* (P_5, H_2)-free graphs that contain an induced ℓK_2 but no clique cutsets. However, the unweighted case is easy:

Theorem 3. *For each fixed positive integer k, the Maximum Stable Set problem can be solved in time* $\mathcal{O}(n^{8k+2})$ *for* (P_5, H_k)*-free graphs.*

Proof. Let H_k^- be the graph obtained from H_k by deleting the degree-1 vertex. We first show that prime (P_5, H_k)-free graphs are H_{2k}^--free.

Let G be a prime (P_5, H_k)-free graph, and suppose that G contains an induced $H = H_{2k}^-$. Let x, y be the two degree-$2k$ vertices in H, and let M be the set of all vertices in G that are adjacent to all vertices in $H \setminus \{x, y\}$. As $x, y \in M$, $|M| \geq 2$. As x, y are nonadjacent, there exists a nontrivial connected component C in $\overline{G}[M]$. As G is prime, there exists a vertex $v \notin C$ distinguishing C. Clearly $v \notin M$, hence there exists a vertex in $H \setminus M$ nonadjacent to v. Moreover, by Observation 1(ii), there are nonadjacent vertices c_1, c_2 in C with $v \sim c_1$, $v \not\sim c_2$.

As G is P_5-free, v may distinguish at most one edge in $H \setminus M$. Hence, as $H \setminus M$ has exactly $2k$ edges, v⓪e holds for at least k edges $e \in H \setminus M$, or v①e holds for at least k edges $e \in H \setminus M$. In the first case, c_1, c_2, v, and k edges e in $H \setminus M$ with v⓪e induce an H_k, a contradiction. In the second case, c_2, v, a vertex in $H \setminus M$ nonadjacent to v, and k edges e in $H \setminus M$ with v①e induce an H_k, a contradiction again. Thus, G must be H_{2k}^--free.

Now, it was shown in [17, Theorem 2] (see also [46]) that for every fixed k, the Maximum Stable Set problem can be solved in time $\mathcal{O}(n^{8k+2})$ for (P_5, H_{2k}^-)-free graphs. Thus, for every fixed k, the Maximum Stable Set problem is solvable within the same running time for prime (P_5, H_k)-free graphs, hence for all (P_5, H_k)-free graphs. □

7. Conclusion

In this paper, we give new applications of the clique separator approach, combine it in one case with modular decomposition and extend some known polynomial time results for the Maximum Weight Stable Set problem. In particular, we have shown that prime atoms of (P_5, F_1)-free graphs are $3K_2$-free and atoms of (P_5, F_2)-free graphs are $4K_2$-free.

As a consequence, the Maximum Weight Stable Set problem is polynomially solvable for (P_5, F_1)-free graphs and for (P_5, F_2)-free graphs, which tremendously generalizes various polynomially solvable cases known before.

We also consider the class of (P_5, H_k)-free graphs which extend both the class of (P_5, F_1)-free graphs and the class of (P_5, F_2)-free graphs. For each fixed k we show that the Maximum Stable Set problem can be solvable in polynomial time for (P_5, H_k)-free graphs.

Open problem. Let H_k^- denote the graph obtained from H_k by deleting the degree-1 vertex in H_k. Is the Maximum *Weight* Stable Set problem solvable in polynomial time for (P_5, H_k^-)-free graphs ($k \geq 2$ fixed)? If yes, the proof of Theorem 3 shows that it is also polynomially solvable for (P_5, H_k)-free graphs, for each fixed positive integer k.

More generally, the following question is of interest: suppose that MS is polynomially solvable for a certain graph class, is MWS solvable in polynomial time on the same graph class, too?

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